Spectral line broadening by relativistic electrons in plasmas: collision operator

A. Naam, S. Douis and M.T. Meftah*
Department of Physics, LRPPS Laboratory, U.K.M. Ouargla University, Ouargla, 30000 Algeria
**E-mail: mewalid@yahoo.com*

May 13 2013

A number of hot astrophysical plasmas, electrons may be energetic enough that their thermal energy kT can be comparable with the rest mass. For the extreme densities encountered in some astrophysical objects, pressure broadening could dominate; however for such objects the electrons may become relativistic due to the extreme temperatures and hence it makes sense to check the modifications to the pressure broadening by relativistic effects. Similarly in laser-produced plasmas very high densities may be achieved with rather modest ion temperatures, resulting in the dominance of Stark over Doppler broadening as well as, in some cases, relativistic electron velocities. In the present work we focus on electron broadening in the impact approximation at first stage, and at the second stage we treat them in collective interaction. We thus revisit the standard semiclassical collision operator and take into account relativistic effects with respect to trajectory and the velocity distribution (Juttner-Maxwell instead of a simple

Maxwellian). We only consider isolated lines, since within the impact approximation the collision operator for more complex cases is basically expressible in terms of this case. This work is organized as follow:

Table of contents

1. Introduction
2. Relativistic collision operator in Coulomb binary interaction
3. Relativistic collision operator in collective interaction
   - description of collective interaction
   - Numerical solution of the integral equation for the collective interaction
   - Electron dynamics (trajectories and microfield autocorrelation function)
   - The collision operator
4. Conclusion

Relativistic collision operator in Coulomb binary interaction

In the relativistic case, we use the standard notation

$$\beta = v/c, \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad (1)$$

To take into account for the deviation from the Maxwellian velocity distribution due to the high electron temperatures, the Maxwell-Juttner distribution:

$$f(\gamma)\, d\gamma = \frac{\gamma^5 \beta^2}{\theta K_2(1/\theta)} e^{-\gamma/\theta} d\beta \qquad (2)$$

is used, where $K_2(x)$ is the modified Bessel function of the second kind and $\theta = k_B T/m_0 c^2$ with m_0 the electron mass ar the rest. Using the polar coordinates, the trajectory in the relativistic case reads:

$$\frac{p^*}{r^*} = 1 + \varepsilon^* \cos \eta\varphi^* \qquad (3)$$

where

$$p^* = \frac{\rho^2\eta^2\gamma}{\rho_0}; \quad \eta = \sqrt{1 - \frac{\rho_0^2\beta^2}{\rho^2\gamma^2}} \tag{4}$$

(we use the star superscript to indicate the relativistic quantities). The relativistic excentricity ε^* is related to the impact parameter ρ, and the demi-axis ρ_0 of the hyperbole by:

$$\varepsilon^* = \sqrt{\frac{\rho^2\eta^2\gamma^2}{\rho_0^2} + 1} \tag{5}$$

by which we can define the relativistic parametric equations as:

$$r^* = \frac{\rho_0}{\gamma}\left(\varepsilon^* \cosh x - 1\right) \tag{6}$$

$$t^* = \frac{\rho_0}{v\gamma^3}\left(\gamma^2\varepsilon^* \sinh x - x\right) \tag{7}$$

which reduce to the non relativistic expressions for $c \to \infty$. The corresponding relativistic expression for the collision operator φ_d^* (which refers to the direct collision operator φ_d [1,2]) is:

$$\varphi_d^*(\omega_1, \omega_2 = -\omega_1) = -\frac{\pi N_e e^2}{\hbar^2} \frac{1}{\theta c^3 K_2(1/\theta)} \int_0^\infty v^3 \gamma^5 \exp(-\frac{mc^2}{k_B T}\gamma) dv \int_{\rho_{\min}}^{\rho_{\max}} \rho d\rho \int_{-\infty}^{\infty} dt_1^* \int_{-\infty}^{t_1^*} dt_2^* e^{\left(i\omega_1 t_1^* - i\omega_1 t_2^*\right)} \left[\overrightarrow{E}^*(t_1^*) \overrightarrow{E}^*(t_2^*) \right]_{ang} \tag{8}$$

Use of the trajectory parametric equations leads to (by inserting the parametric equations of the relativistic electron trajectory (6), (7) in the expression (8):

[1] S. Alexiou, Phys. Rev. A 49, 106. (1994)

[2] s. Sahal-Bréchot, Astron, Astrophys. 2, 322 (1969)

$$\varphi_d^*(0,0) = -\frac{\pi N_e e^4}{3\hbar^2}\frac{1}{(4\pi\varepsilon_0)^2}\frac{1}{\theta c^3 K_2(1/\theta)}\int_0^\infty \frac{v\gamma^3}{\rho_0^2}\exp(-\frac{mc^2}{k_B T}\gamma)dv \int_{\rho_{\min}}^{\rho_{\max}} \rho d\rho \int_{-\infty}^{\infty}(\varepsilon^*\cosh x_1 - 1)\int_{-\infty}^{x_1}(\varepsilon_2^* - 1)\,dx_1 dx_2 \frac{(\varepsilon^* - \cosh x_1)(\varepsilon^* - \cosh x_2) + \left(\varepsilon^{*2} - 1\right)\sinh x_1 \sinh x_2}{(\varepsilon^*\cosh x_1 - 1)^3(\varepsilon_2^* - 1)^3} \quad (9)$$

By using the formula (5) we transform the integration over ϱ in (9) by the integration over ε^*:

$$\rho d\rho = \frac{\rho_0^2\varepsilon^*}{\gamma^2}d\varepsilon^* \quad (10)$$

the interference term becomes then:

$$\varphi_d^*(0,0) = -\frac{\pi N_e e^4}{3\hbar^2}\frac{1}{(4\pi\varepsilon_0)^2}\frac{1}{\theta c^3 K_2(1/\theta)}\int_0^\infty v\gamma\exp(-\frac{mc^2}{k_B T}\gamma)dv \int_{\varepsilon^*_{min}}^{\varepsilon^*_{max}}\frac{d\varepsilon^*}{\varepsilon^*}G_1^{*2}(0,\varepsilon^*) \quad (11)$$

where

$$G_1^*(0,\varepsilon^*) = \int_{-\infty}^{\infty} dx \frac{\left(\gamma^2\cosh x - \frac{1}{\varepsilon^*}\right)\left(1-\frac{\cosh x}{\varepsilon^*}\right)}{\left(\cosh x - \frac{1}{\varepsilon^*}\right)^3} \quad (12)$$

by integrating over ε^*, the expression of $\varphi_d^*(0,0)$ becomes as:

$$Re\varphi_d^*(0,0) =$$

$$-\frac{\pi N_e e^4}{3\hbar^2}\frac{1}{(4\pi\varepsilon_0)^2}\frac{1}{\theta c^3 K_2(1/\theta)} Re\left(\int_0^\infty \gamma v dv \exp(-\frac{mc^2}{k_B T}\gamma)\right.$$

$$\left[4\gamma^4 \ln\frac{\varepsilon^*_{\max}}{\varepsilon^*_{\max}} - \frac{\left(\gamma^2-1\right)^2}{4\left(\varepsilon^{*2}_{\max}-1\right)} - \frac{\varepsilon^{*4}_{\max}\left(\gamma^2-1\right)^2}{\left(\varepsilon^{*2}_{\max}-1\right)^2}\arctan^2\sqrt{\frac{\varepsilon^*_{\max}+1}{\varepsilon^*_{\max}-1}}\right.$$

$$+\arctan\sqrt{\frac{\varepsilon^*_{\max}+1}{\varepsilon^*_{\max}-1}}\left(\frac{6-7\varepsilon^{*2}_{\max}}{\left(\varepsilon^{*2}_{\max}-1\right)^{3/2}}\left(\gamma^2-1\right)^2 - \frac{8\left(\gamma^2-1\right)}{\sqrt{\varepsilon^{*2}_{\max}-1}}\right)$$

$$+\frac{\left(\gamma^2-1\right)^2}{4\left(\varepsilon^{*2}_{\min}-1\right)} + \frac{\varepsilon^{*4}_{\min}\left(\gamma^2-1\right)^2}{\left(\varepsilon^{*2}_{\min}-1\right)^2}\arctan^2\sqrt{\frac{\varepsilon^*_{\min}+1}{\varepsilon^*_{\min}-1}}$$

$$\left.\left.-\arctan\sqrt{\frac{\varepsilon^*_{\min}+1}{\varepsilon^*_{\min}-1}}\left(\frac{6-7\varepsilon^{*2}_{\min}}{\left(\varepsilon^{*2}_{\min}-1\right)^{3/2}}\left(\gamma^2-1\right)^2 - \frac{8\left(\gamma^2-1\right)}{\sqrt{\varepsilon^{*2}_{\min}-1}}\right)\right]\right)$$

where γ is given by the formula (1), $\varepsilon^*_{\text{min}}$, $\varepsilon^*_{\text{max}}$ are given by using(5) as:

$$\varepsilon^*_{\text{max}} = \sqrt{\frac{\rho^2_{\text{max}}(1 - \frac{\rho_0^2\beta^2}{\rho^2_{\text{max}}\gamma^2})\gamma^2}{\rho_0^2} + 1} \tag{13}$$

$$\varepsilon^*_{\text{min}} = \sqrt{\frac{\rho^2_{\text{min}}(1 - \frac{\rho_0^2\beta^2}{\rho^2_{\text{min}}\gamma^2})\gamma^2}{\rho_0^2} + 1} \tag{14}$$

When we make the limit $c \rightarrow \infty$ the relativistic term $\varphi^*_d(0,0)$ goes to the non-relativistic term $\varphi_d(0,0)$ defined by [1]

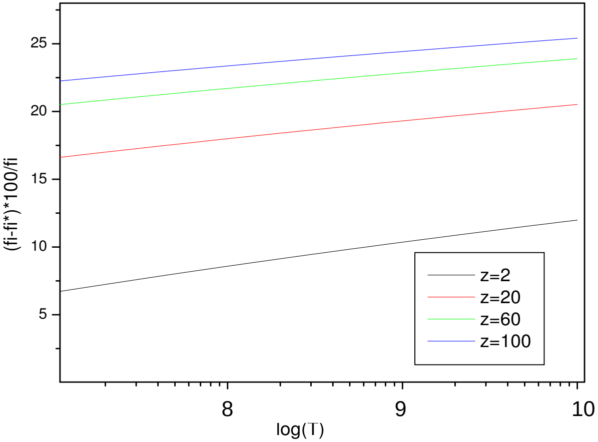

Figure 1: Comparison between classical and relativistic collision operator

description of collective interaction

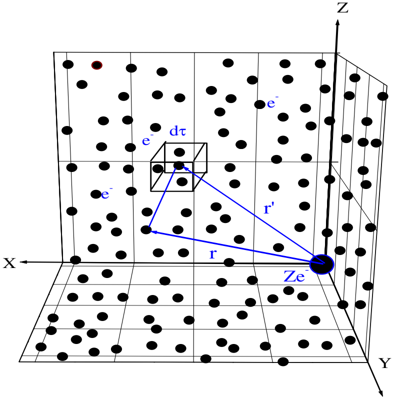

Figure 2: An impurity immersed in a plasma

$$V(r) = V_{ie}(r) + V_{ee}(r) + V_{ef}(r) \tag{15}$$

- V_{ie}interaction electron-ion= Kelbg
- V_{ee}interaction electron-electron= Debye
- V_{ef}interaction electron-background= Kelbg

$$V_{ie}(r) = -\frac{Ze^2}{r}(1 - e^{-(r/(\sqrt{\pi}\lambda_T))^2} + \frac{r}{\lambda_T}(1 - erf(\frac{r}{\sqrt{\pi}\lambda_T}))) \tag{16}$$

Using spherical coordinates and some basic calculations, the equation (15) is transformed into the following:

$$V(r) = V_{ie}(r) - 2\pi n_e e^2 \lambda_D \int_0^\infty \frac{r'}{r}\left(e^{-\frac{|r+r'|}{\lambda_D}} - e^{-\frac{|r-r'|}{\lambda_D}}\right) e^{-\beta V(r')} dr'$$
$$+2\pi n_e e^2 \frac{1}{\lambda_T}\int_0^\infty \frac{r'}{r}\left[G(r+r') - G\left(\left|r-r'\right|\right)\right] dr' \quad (17)$$

where

$$G(r) = \frac{r}{2}[-r(2\lambda_T + r + \lambda_T e^{-(\frac{r}{\sqrt{\pi}\lambda_T})^2}) + \pi\lambda_T^2 erf(\frac{r}{\sqrt{\pi}\lambda_T})(\frac{3}{2} + (\frac{r}{\sqrt{\pi}\lambda_T})^2)]$$

Numerical solution of the integral equation for the collective interactionthe

The numerical solution of the nonlinear integral equations (17), in the case $\Gamma = e^2/(aK_B T) = 0.1$ (where $a = (3/4\pi n_e)^{1/3}$), $\eta = \lambda_T/a = 0.177$, $\eta' = \lambda_D/a = 1.826$ and $Z = 2$ and $Z = 8$ by the fixed point method (FPM) gives the effective potential energy :

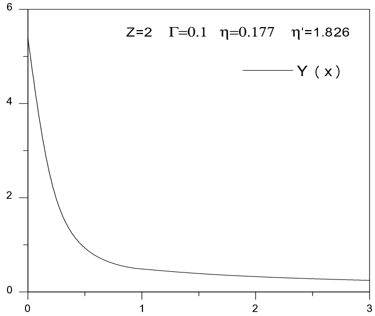

Figure 3: Effective potential energy of the electronfor Z=2

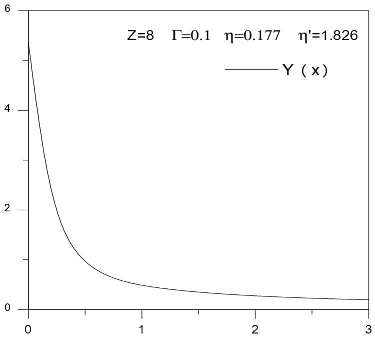

Figure 4: Effective potential energy of the electron for Z=8

Electron dynamics (trajectories and microfield autocorrelation function)

The calculation of the real trajectories of relativistic electrons in a hot plasma, is a necessary step to calculate several dynamic properties such as the time autocorrelation function, the diffusion coefficient and the electric permittivity...
The relativistic force acting on an electron in the plasma is equal to the derivative of the quantity of motion:

$$\overrightarrow{F} = \frac{d\overrightarrow{P}}{dt} = \frac{dm}{dt}\overrightarrow{v} + m\frac{d\overrightarrow{v}}{dt} \tag{18}$$

therefore this force is equal to:

$$\overrightarrow{F} = \frac{m_0}{c^2}\left(1 - v^2/c^2\right)^{-3/2}\left(\overrightarrow{v}\cdot\overrightarrow{\Gamma}\right)\overrightarrow{v} + \frac{m_0}{\sqrt{1 - v^2/c^2}}\overrightarrow{\Gamma} \tag{19}$$

and the force on the other hand equal:

$$\overrightarrow{F}=\frac{d\overrightarrow{P}}{dt}=-\overrightarrow{\nabla}V(r) \tag{20}$$

where $V(r)$ represents the potential energy (17) of an electron plasma at position r of origin. We equate the equations (19) and

(20) member to member, we find the equation:

$$\frac{m_0}{c^2}\left(1-v^2/c^2\right)^{-3/2}\left(\overrightarrow{v}\cdot\overrightarrow{\Gamma}\right)\overrightarrow{v}+\frac{m_0}{\sqrt{1-v^2/c^2}}\overrightarrow{\Gamma}=-\overrightarrow{\nabla}V(r) \tag{21}$$

The total electric microfield due to the electrons on the impurity centered at the coordinates origin is given by:

$$\overrightarrow{E} = \sum_{k=1}^{N_e} \overrightarrow{e_{ie}}(r_k) \tag{22}$$

$$C_{EE}(t) = \frac{a^4}{e^2} \left\langle \overrightarrow{E}(t) \cdot \overrightarrow{E} \right\rangle \tag{23}$$

$$C_{EE}(t) = \frac{a^4}{e^2} \int f\left(\overrightarrow{r}, \overrightarrow{v}\right) \overrightarrow{e}\left(r\right) \cdot \overrightarrow{e}_{mf}(r(t)) d\overrightarrow{r} d\overrightarrow{v} \qquad (24)$$

where:

$$\overrightarrow{e}_{mf}(\overrightarrow{r}) = \frac{1}{Ze} \overrightarrow{\nabla} V(r) \qquad (25)$$

and $f(r, v)$ is the Maxwell-Juttner-Boltzmann distribution given by:

$$f(r, v) = exp(-(mc^2 + V(r))/kT)/(m_e^3 c^3 K_2(m_0 c^2/kT)) \qquad (26)$$

here $m = m_0 \gamma$ and $K_2(x)$ is the modified Bessel function.
where $\overrightarrow{r}(t)$ is the time-dependent position vector. We get it at all time t, we have solved numerically (using Verlet algorithm)the movement equation $d\overrightarrow{P}/dt = -e.\overrightarrow{e}_{mf}(r)$ with $\overrightarrow{P} = m_0 \overrightarrow{v}/\sqrt{(1 - v^2/c^2)}$ is the relativistic momentum of the electron. In the calculation of $C_{EE}(t)$ the average on the velocities is done on the relativistic distribution $f(r, v)$.

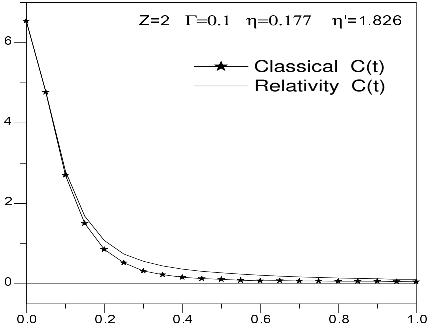

Figure 5: Classical and relativistic electric field auto-correlation function for Z=2

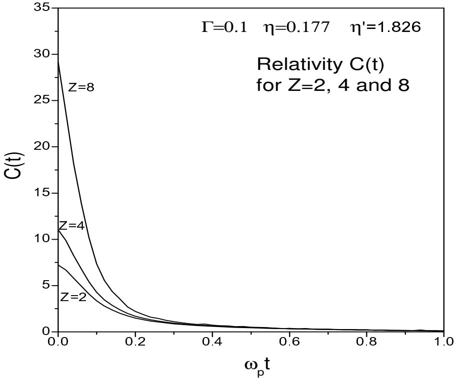

Figure6: Relativistic electric field auto-correlation function for different Z

Regarding the function $C_{EE}(t)$ (5) , we found:
- When you move away from $t = 0$, the relativistic effect is manifested more clearly.
- In Figrs (5) and (6) we note that when we increase the charge number Z the covariance $C(0)$ also increases, but the relativistic curves decrease more quickly (vanished quickly and never cut the Ox axis).

The collision operator

$$C_{EE}(t_1 - t_2) = < \overrightarrow{E}(t_1).\overrightarrow{E}(t_2) >_{can} \tag{27}$$

we can check that:

$$\phi_d = -\frac{e^4}{3\hbar^2 a^4} \int_0^{\infty} C_{EE}(t)dt \tag{28}$$

$$\phi = -\overrightarrow{R_n}.\overrightarrow{R_n}(\frac{e^4}{3.\hbar^2 a^4}) \int_0^{+\infty} C_{EE}(t)dt \equiv \overrightarrow{R_n}.\overrightarrow{R_n}\phi_d \tag{29}$$

In computing the collision operator ϕ, the plasma electron (the perturber) moves in the effective field created by all the plasma. Moreover this electron creates a field(Kelbg) at the impurity ion. Then we call ϕ_d the amplitude of the collision operator because it is this quantity that contains the plasma parameters through the correlation function $C_{EE}(t)$. This contains all the information regarding the density, the temperature and the charge number Z of ions. We present on a table the ratio of the amplitudes of the collision operator between the classical and relativistic case for different plasma conditions such as n_e, T and Z. We find that when Z is increased the relativistic effect increases.

Z	$\phi_d^{classical} / \phi_d^{relativistic}$
1	0.9154
2	0.8263
3	0.6554
4	0.3396

Conclusion

In this work, we have calculated the amplitude of the collision operator. This calculation takes into an account the relativistic mass of the free plasma electrons in the description of the dynamics. At first, we have assumed that the relativistic free electron moves in the Coulomb potential due to an impurity ion with a net charge (+Ze) and at the second stage, the free relativistic electrons are assumed to move in the effective potential that we have calculated in the mean field approximation. In both cases, the amplitude of the collision operator is calculated.