Shape Modelling with Family of Pearson Distributions

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Shape modelling

- The choice of the best-suited statistical distribution for data modelling is not a trivial issue;
- Unless a sound theoretical background exists for selecting a particular distribution, one will usually try to test various candidates and select a distribution based on its fit to the observed data;
- It is more efficient to define a sufficiently general family that can be used for this purpose.

Pearson system - great diversity of shapes:

- unimodal, bimodal, U-shaped, J-shaped and monotone probability distribution functions,
- ...which may be symmetric and asymmetric, concave and convex,
- ...with smooth, abrupt, truncated, long, medium or short tails.

Pearson system*)

- First derivative of probability density function:

$$\frac{1}{f(x)}\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\frac{a+x}{c_0 + c_1 x + c_2 x^2}$$

- Asymmetry ($As^2 = \beta_1$)

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2}$$

- Excess (β_2)

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

Using only 2 parameters: Squared Asymmetry (β_1) and Excess (β_2), calculated from observations, Type of Pearson distribution can be retrieved.

*) Pearson, K.: 1895, *Contributions to the Mathematical Theory of Evolution. II. Skew Variation in Homogeneous Material.* Philosophical Transactions of the Royal Society of London, **186**, 343 – 414

Method of moments

$$c_0 = (4\beta_2 - 3\beta_1)(10\beta_2 - 12\beta_1 - 18)^{-1}\mu_2$$

$$a = c_1 = \sqrt{\beta_1}(\beta_2 + 3)(10\beta_2 - 12\beta_1 - 18)^{-1}\sqrt{\mu_2}$$

$$c_2 = (2\beta_2 - 3\beta_1 - 6)(10\beta_2 - 12\beta_1 - 18)^{-1}$$

$$\kappa = \frac{1}{4}c_1^2(c_0c_2)^{-1} = \frac{1}{4}\beta_1(\beta_2 + 3)^2(4\beta_2 - 3\beta_1)^{-1}(2\beta_1 - 6)^{-1}$$

Classification

I: $\kappa < 0$

II: $\beta_1 = 0,\ \beta_2 < 3$

III: $2\beta_2 - 3\beta_1 - 6 = 0$

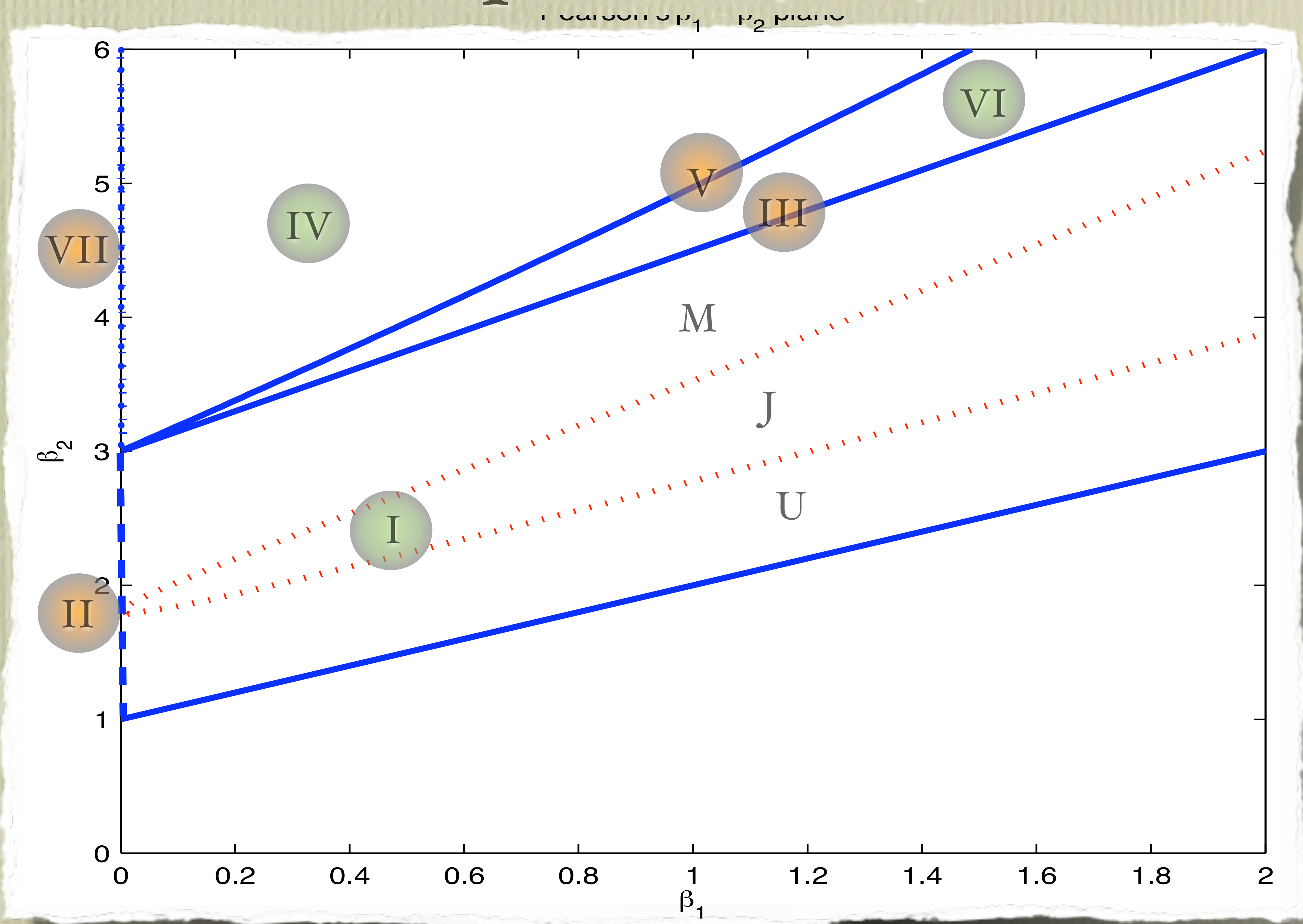
IV: $0 < \kappa < 1$

V: $\kappa = 1$

VI: $\kappa > 1$

VII: $\beta_1 = 0,\ \beta_2 > 3$

Beta plane (β_1, β_2)



Excess β_2

Square of Asymmetry β_1

Method of Maximum Likelihood

- **The idea[*]:** to find the parameters of probability density function that give the highest probability (maximum likelihood) of the occurrence of the measured data.

[*] Sir Ronald Aylmer Fisher (1890-1962) for the first time presented the idea in 1912 (when he was 22 years old) in the article: *On an absolute criterion for fitting frequency curves*, Messenger of Mathematics (1912), **41,** 155-160.

Method of Maximum Likelihood

- **Probability:**

  Knowing parameters
  $\rightarrow$ Prediction of outcome

- **Likelihood:**

  Observation of data
  $\rightarrow$ Estimation of parameters

$$f(\mathbf{x}|\boldsymbol{\theta}) \equiv L(\boldsymbol{\theta}|\mathbf{x})$$

probability density f-on

likelihood function

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$$

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)^{\mathrm{T}}$$
$$\boldsymbol{\theta} = (a, c_0, c_1, c_2)^{\mathrm{T}}$$

Likelihood function

$$L(\boldsymbol{\theta}|\mathbf{x}) \equiv f(\mathbf{x}|\boldsymbol{\theta}) = \prod_{i=1}^{n} f_i(x_i|\boldsymbol{\theta})$$

applying logarithm, one obtain:

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{x}) = \ln L(\boldsymbol{\theta}|\mathbf{x}) = \sum_{i=1}^{n} \ln f_i(x_i|\boldsymbol{\theta})$$

Looking for $\boldsymbol{\theta}^*$

- Looking for $\boldsymbol{\theta}^*$ which maximizes likelihood

$$\mathcal{L}(\boldsymbol{\theta}^*|\mathbf{x}) = \max_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}|\mathbf{x}) = \max_{\boldsymbol{\theta}} \sum_{i=1}^{n} \ln f_i(x_i|\boldsymbol{\theta})$$

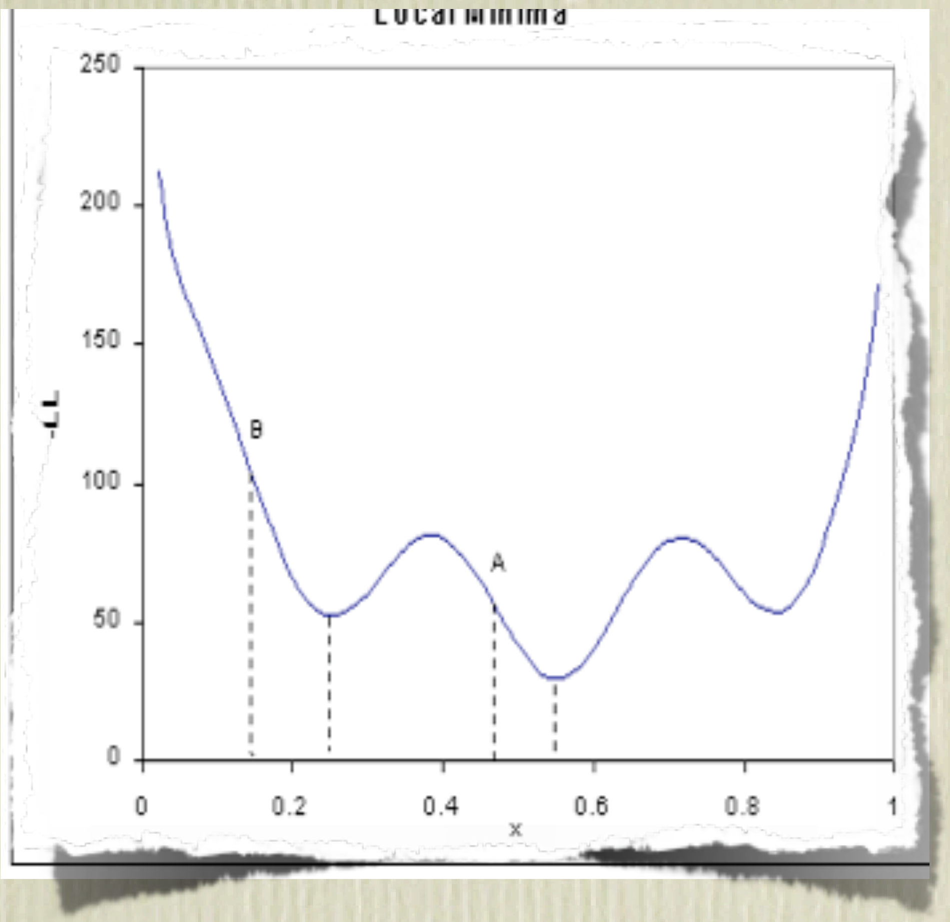
- It is not possible to solve this task analytically, thus, we apply numerical methods of optimization.

Numerical optimization

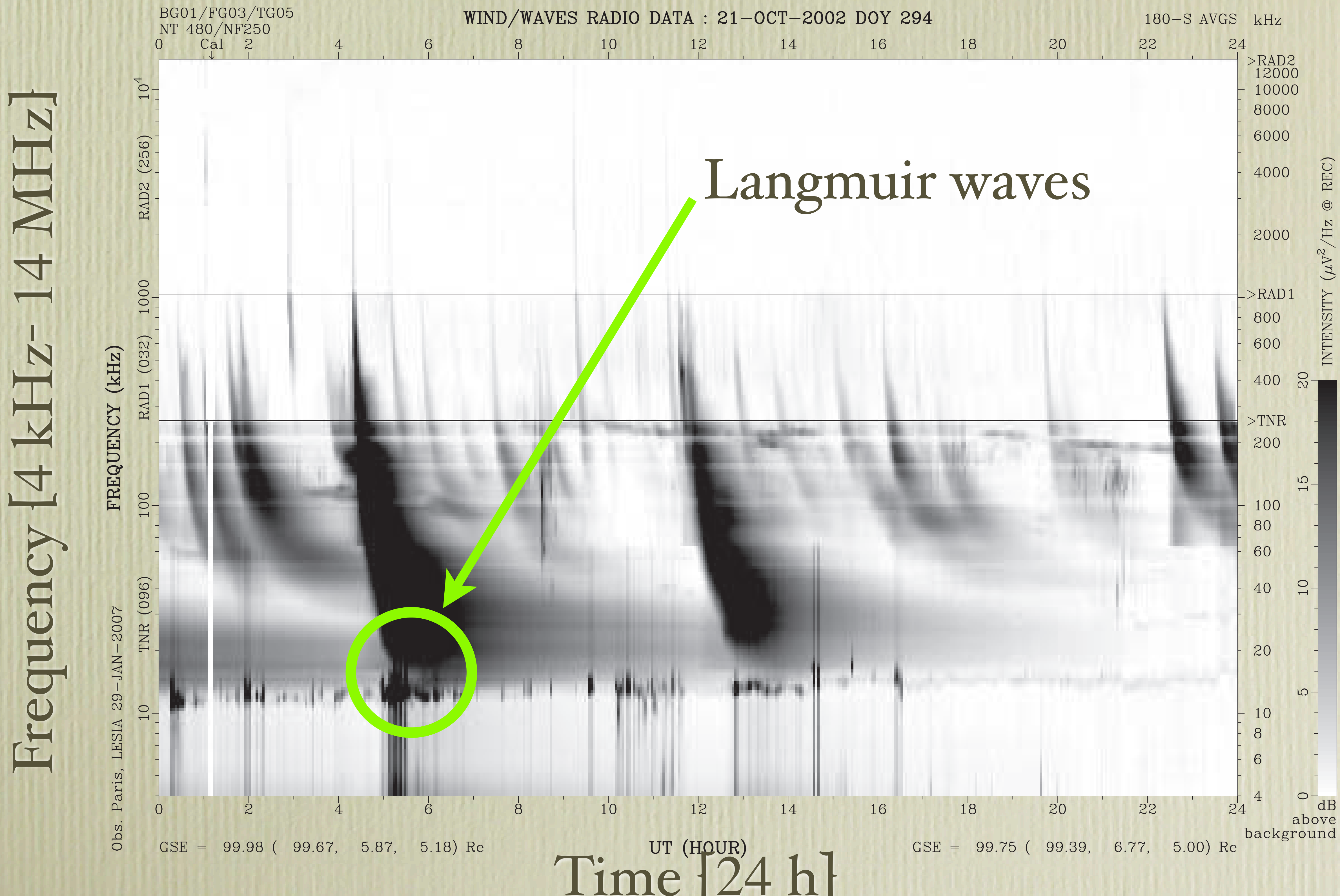
- Methods, e.g.:
  - Nelder - Mead
  - Levenberg - Marquardt
- It is important to choose GOOD starting values for the parameters

  They can be calculated from observations using method of moments!

1 D example



Observations, dynamical spectrum



A theoretical prediction

- Stochastic Growth Theory (SGT)*)

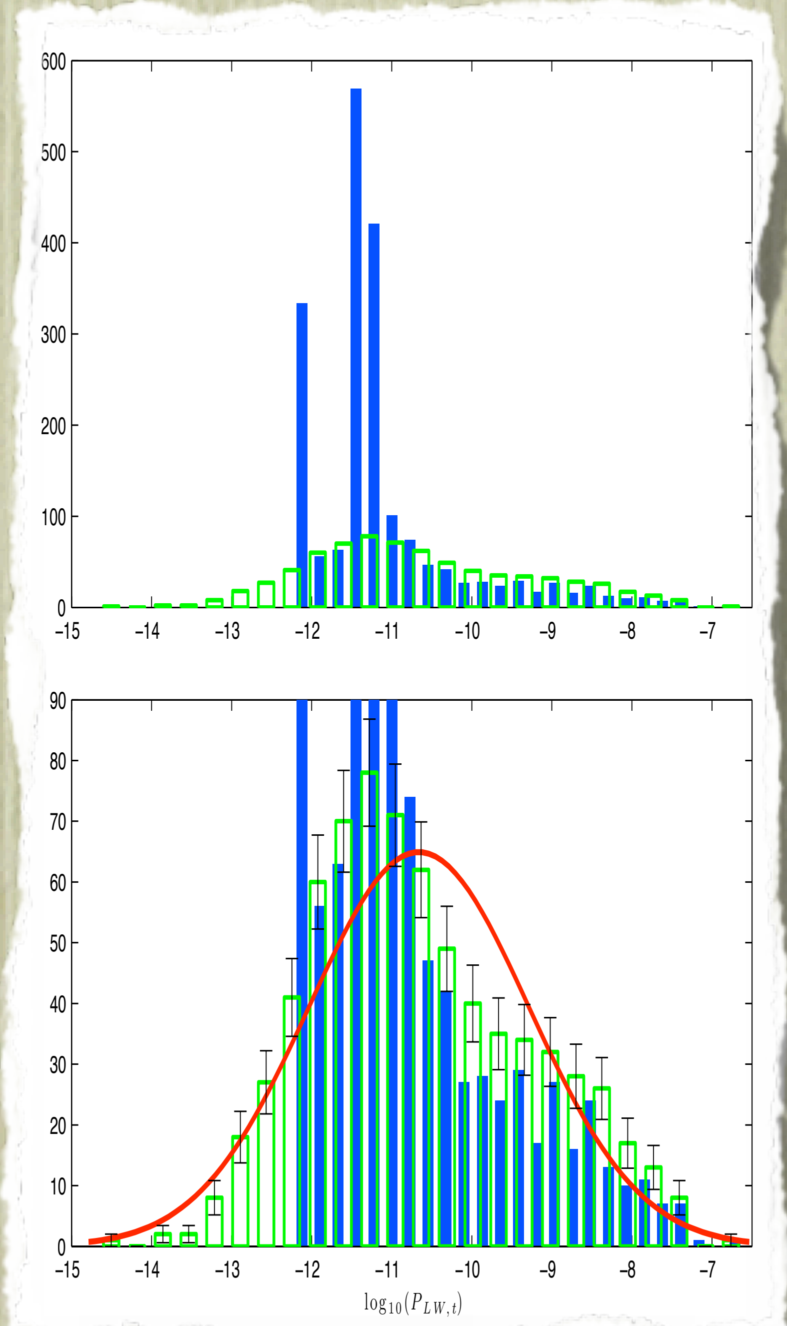
$$G = 2\log\left(\frac{E}{E_0}\right)$$

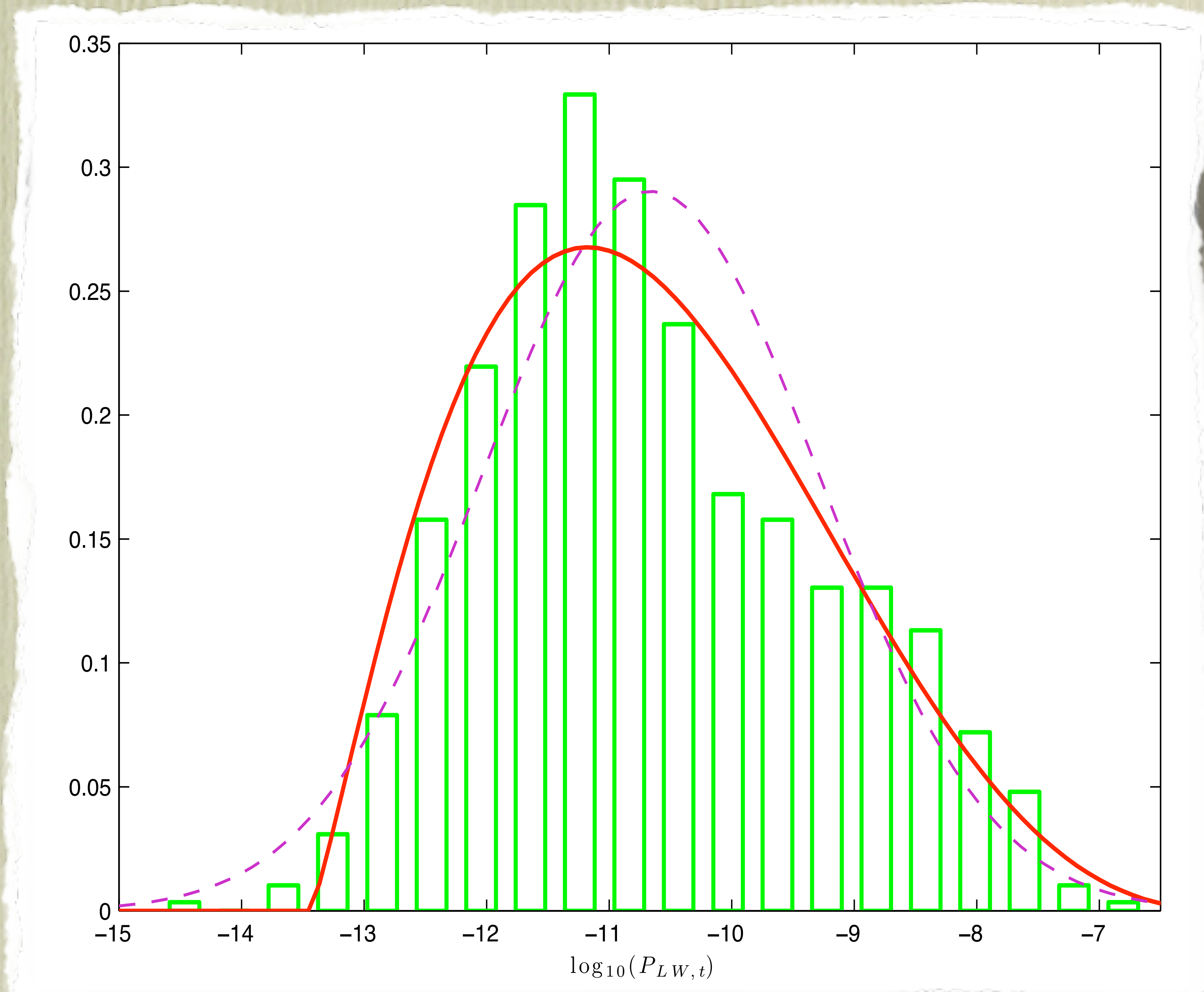
$$\log E = \log E_0 + \sum_{i=1}^{N} G_i \qquad N \gg 1$$

- Central limit theorem of statistics says: if the process of energy exchange (log *E*) has a random character and the number of these exchanges is large enough, then the probability density distribution of the electric field measurements is NORMAL.

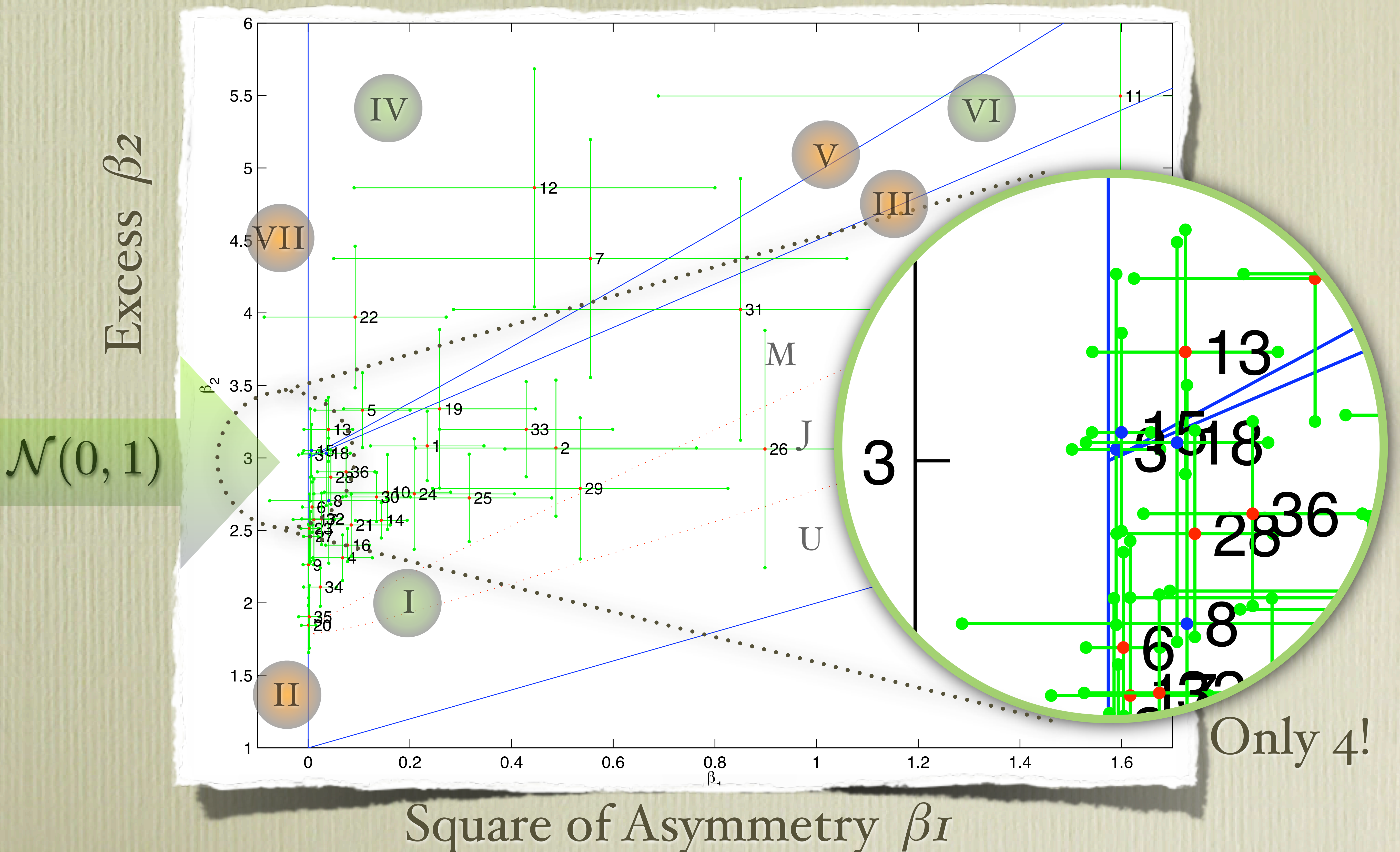
*) Robinson, P. A. : *Stochastic-Growth Theory of Langmuir growth-rate fluctuations in type III solar radio sources,* 1993, Solar Physics, **146,** 357

Shape modelling





36 LW events in Beta plane (β_1, β_2)



Resume

- LW distribution seems to be more Pearson type than normal - in contradiction with SGT!
- REOPENED QUESTIONS:
    - What is distributions of Langmuir waves energy?
    - Which plasma processes lead to the observed LW energy distribution?