Poster

THE EQUATION OF STATE OF A DEGENERATE FERMI GAS

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Abstract. An analytical expression for Fermi-Dirac integrals of arbitrary order is presented, and its applicability in obtaining an analytical EOS of a degenerate non-relativistic Fermi gas is discussed to some extent.

1. INTRODUCTION

Degenerate Fermi gases occur in a variety of systems studied in astrophysics and "laboratory" physics. Examples of such systems include white dwarfs and neutron stars, planetary interiors, ordinary metals and organic conductors.

The general form of the equation of state (EOS) of a non-relativistic degenerate Fermi gas has been determined more than half a century ago (for example, Chandrasekhar, 1939) as:

$$n_e = \frac{4\pi}{h^3}(2m_e k_B T)^{3/2} F_{1/2}(\eta) \tag{1}$$

All the symbols have their standardized meanings, and $F_{1/2}(\eta)$ denotes a particular case of the Fermi-Dirac (FD) integrals of order n:

$$F_n(\eta) = \int_0^\infty \frac{f(\epsilon)d\epsilon}{1 + \exp[\beta(\epsilon - \mu)]} \tag{2}$$

where $f(\epsilon) = \epsilon^n$, $n \in \mathbb{R}$, μ denotes the chemical potential, β is the inverse temperature, and $\eta = \beta\mu$.

The practical problem of the numerical evaluation of the FD integrals has been the object of several recent studies (such as Cloutman, 1989; Antia, 1993; Miralles and Van Riper, 1995). The purpose of the present paper is to obtain an analytical approximation to the FD integrals of arbitrary order, and to apply it to the problem of the EOS of a degenerate Fermi gas.

2. THE CALCULATIONS

Taking $k_B = 1$, and introducing a change of variables by

$$\epsilon - \mu = Tz \tag{3}$$

eq. (2) can be transformed into the following form:

$$F_n(\beta\mu) = T \int_{-\mu/T}^{\infty} \frac{f(\mu + Tz)}{1 + \exp[z]} = \int_0^{\mu} f(\epsilon)d\epsilon + T \int_0^{\infty} \frac{f(\mu + Tz) - f(\mu - Tz)}{1 + \exp[z]} dz \quad (4)$$

The difference of values of f in the last integral of eq. (4) can be developed into series as

$$f(\mu + Tz) - f(\mu - Tz) = \sum_{n=0}^{\infty} [1 - (-1)^n] \frac{(Tz)^n}{n!} f^{(n)}(\mu) \quad (5)$$

Inserting eq. (5) into eq. (4) and using the fact that

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1 + \exp[z]} dx = (1 - 2^{1-\alpha})\Gamma(\alpha)\zeta(\alpha)$$

where $\Gamma(\alpha)$ and $\zeta(\alpha)$ denote the gamma function and Riemann's zeta function, one gets the following final form of the FD integrals of arbitrary order:

$$F_n(\beta\mu) = \int_0^{\mu} f(\epsilon)d\epsilon + T \sum_{n=0}^{\infty} \frac{f^{(n)}(\mu)}{n!} [1 - (-1)^n] T^n (1 - 2^{-n}) \Gamma(n+1)\zeta(n+1) \quad (6)$$

This expression is a generalization to arbitrary order and number of terms of existing partial results (for example, Landau and Lifchitz, 1976).

3. DISCUSSION

Inserting $n = 1/2$ into eq. (6) and limiting the sum to terms up to and including T^6, it follows that

$$F_{1/2}(\eta) \cong \frac{2}{3}\mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{T}{\mu}\right)^2 + \frac{7\pi^4}{640} \left(\frac{T}{\mu}\right)^4 + \frac{31\pi}{3072} \left(\frac{T}{\mu}\right)^6 \right] \quad (7)$$

This result is impractical, because the chemical potential itself is a function of T and the number density of the Fermi gas. It can be shown from eq. (1) that this function can be approximated by the following developement:

$$\mu \cong \mu_0 \left[1 - \frac{\pi^2}{12} \left(\frac{T}{\mu_0}\right)^2 + \frac{\pi^4}{720} \left(\frac{T}{\mu_0}\right)^4 - \frac{\pi^6}{162} \left(\frac{T}{\mu_0}\right)^6 + \frac{\pi^8}{754} \left(\frac{T}{\mu_0}\right)^8 \right] \quad (8)$$

The symbol μ_0 denotes the chemical potential of the electron gas at T = 0 K, and it depends only on the particle number density. Inserting eq. (8) into eq. (7) and developing into series in T, one gets

$$F_{1/2}(T) \cong \frac{2}{3}\mu_0^{3/2} \left[1 + \frac{\pi^4}{48} \left(\frac{T}{\mu_0}\right)^4 + \frac{19\pi^6}{5760} \left(\frac{T}{\mu_0}\right)^6 + \frac{\pi^8}{146} \left(\frac{T}{\mu_0}\right)^8 \right] \quad (9)$$

Note that we have managed to express this FD integral as an explicite function of the particle number density and the temperature. This result is a distnct advantage over some previous numerical work (such as Cloutman, 1989), where the argument of this integral was left in the form $\beta\mu$, without taking into account the number-density and temperature dependences of the chemical potential.

How can eq. (9) be applied to in obtaining an analytical EOS of a degenerate Fermi gas? Inserting, at first, the known result for μ_0 into eq. (9), one would get an expression showing explicitely the dependence of a FD integral of the order 1/2 on the particle number density and temperature. Going a step further and inserting this result into eq. (1), one would get the required EOS, in the form of an equation relating the number density, temperature and various known constants (such as h and k_B). This result has the form

$$n_e - K_1 W n_e T^{3/2} - K_2 W n_e^{-5/3} T^{11/2} - K_3 W n_e^{-3} T^{15/2} - K_4 W n_e^{-13/3} T^{19/2} = 0 \quad (10)$$

where n_e denotes the number density of the electrons and all the other symbols denote different combinations of known constants. The low temperature limit of this equation, obtained by developing it into series in T up to terms including T^9, is solvable within S. Wolfram's MATHEMATICA 3 software package in a few minutes on a Pentium MMX/166 with 32 Mbytes RAM. The solution has the form

$$n \cong -\frac{1}{6^{1/4}}[[\frac{6K_3 W T^{15/2} - 12K_1 K_3 W^2 T^9 + 6K_1^2 K_3 W^3 T^{21/2}}{1 - 3K_1 W T^{3/2} + 3K_1^2 T^3 - K_1^3 W^3 T^{9/2}} - << \ldots >> \quad (11)$$

where $<< .. >>$ denotes terms omitted due to space limitations.

We have thus obtained an analytical form of the EOS of a degenerate Fermi gas. The volume per particle, needed in some applications, is simply the inverse density. Our result is approximate, in the sense that the developement in eq. (9) is limited to a small number of terms. Increasing the number of terms and applying this EOS to real physical systems will be the subject of future work.

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References

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