VI Serbian-Belarusian Symp. on Phys. and Diagn. of Lab. &
Astrophys. Plasma, Belgrade, Serbia, 22 - 25 August 2006
eds. M. Ćuk, M.S. Dimitrijević, J. Purić, N. Milovanović
Publ. Astron. Obs. Belgrade No. 82 (2007), 261-268 *Contributed paper*

A METHOD FOR DETERMINING ORBITS OF SMALL PLANETS AND THEIR 3D REPRESENTATION

N. Pejović, M. Marić, Ž. Mijajlović

Faculty of Mathematics, University of Belgrade, Belgrade, Serbia

Abstract. We propose a method for determining elements of orbits of small planets and comets from data obtained by series of measurements. The partic ular benefit of the method is seen in the easy and accurate determination of the type of the orbit when the measured points are closely arranged on the trajectory of the body, i.e. the time intervals between observations are small. It is done by computing four quantities related to conics that represent trajectories. Also, software is developed for the 3D graphical representation of trajectories and enveloping surfaces that they make in the course of time in the heliocentric coordinate system.

1. INTRODUCTION

As it is well known, the computing of elements of the orbit of a small celestial body for the given two heliocentric positions r_1, r_2 in moments t_1, t_2 is simple if the parameter p of the orbit is known. The problem of finding the parameter p is solved by Gauss (Theoria motus corporum coelestium, 1809) and it is still basic for contemporary computing. In this computation appears a quantity x, the solution of the system of equations:

$$\eta^3 - \eta^2 = mX(x), \quad x = m\eta^{-2} - l.$$

Then the orbit of the body is: elliptic, if $x > 0$; parabolic, if $x = 0$; hyperbolic if $x < 0$.

Now, suppose that we have a series of measurements of positions of a celestial body. The result is the cloud of points, i.e their coordinates that represent the trajectory of the body with certain precision. We propose an algorithm and it's implementation for finding the algebraic equation representing the trajectory. In fact we did more, we proposed the method for finding the equation of the second order surface *S* from a set *SP* of measured points. Then, the trajectory, a second order algebraic curve is obtained simply choosing z = 0 for an appropriately chosen coordinate system.

The classification of second order surfaces is the key part of the method. We use in the classification two matrices of small order having as elements coefficients of the surface. i.e. it's equation. The surface is determined according two the number of their eigenvalues and their signs. This approach permits simple implementation and good results in recognition of the surface, in particular when the points from the set P are concentrated in the small area on the surface S. The recognition of the surface is performed in the three stages. The first step consists of the finding an approximate equation of the surface, in the second, eigenvalues of matrices are determined, and finally, the type of the surface is concluded. The determination of the equation is done by solving of a system of linear equations, where the method of singular decomposition of matrices is used. Eigenvalues of matrices are computed by the Jacoby method. The algorithm in the details is as follows.

The second order surfaces are presented by the equation:

$$f(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2px + 2qy + 2rz + d \quad (1)$$

where $a^2 + b^2 + c^2 + f^2 + g^2 + h^2 \neq 0$

Another form of this equation is:

$$f(x,y,z) = \langle Mr, r\rangle + 2\langle n, r\rangle + d. \quad (2)$$

where r = [x, y, z]^T is the position vector in R^3, *u, v* denotes the scalar product of vectors *u, v*, and

$$M = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \quad n = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

The advantage of the method we are proposing is that it gives good results when collected data are concentrated in the small area of the supposed surface. This is achieved by introducing the matrix *N*, usually not considered in other methods, and to which the following theorem refers. This theorem has the crucial role in determination of the type of the surface.

Theorem: Let

$$N = \begin{bmatrix} a & h & g & p \\ h & b & f & q \\ q & f & c & r \\ p & q & r & d \end{bmatrix} \quad (3)$$

and *r3 = rank M, r4= rank N and* δ *=det(N)*. Further, *M-sign* and *N-sign* denote the signs of eigenvectors of matrices *M* and *N* respectively. Then the corresponding surface is:

r3	***r4***	δ	***M-sign***	***N-sign***	***Surface***
3	4	<0			Ellipsoid
3	4	>0	different		One-sheet hyperboloid
3	4	<0	different		Two-sheet hyperboloid
3	3		different		Conus
2	4	<0	same		Elliptical paraboloid
2	4	>0	different		Hyperbolical paraboloid
2	3		same	different	Elliptical cylinder
2	3		different		Hyperbolical cylinder
2	2		different		Two intersecting planes
1	3				Parabolical cylinder
1	2			different	Two parallel planes
1	1				Two coincident planes

Imaginary surfaces and degenerate cases (points) are omitted from the table, although they are characterized also by values *r3, r4, δ* and *M-sign* and *N-sign*. The proof of the theorem is long, and we omit it from this presentation.

2. THE ALGORITHM

Suppose that the set P of measurements consists of m points. Instead of determining the equation (1), we are computing the normalized form:

$$k_1 x_2 + k_2 y_2 + k_3 z_2 + 2k_4 yz + 2k_5 xz + 2k_6 xy + 2k_7 x + 2k_8 y + 2k_9 z = -1 \quad (4)$$

having one unknown less. Here $x_2 = x^2$, $y_2 = y^2$ and $z_2 = z^2$. We form the matrix S of measured positions of dimension m × 9,

$$S = \begin{pmatrix} x_2^1 & y_2^1 & z_2^1 & 2y^1z^1 & 2x^1z^1 & 2x^1y^1 & 2x^1 & 2y^1 & 2z^1 \\ x_2^2 & y_2^2 & z_2^2 & 2y^2z^2 & 2x^2z^2 & 2x^2y^2 & 2x^2 & 2y^2 & 2z^2 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ x_2^{10} & y_2^{10} & z_2^{10} & 2y^{10}z^{10} & 2x^{10}z^{10} & 2x^{10}y^{10} & 2x^{10} & 2y^{10} & 2z^{10} \\ . & . & . & . & . & . & . & . & . \\ x_2^m & y_2^m & z_2^m & 2y^mz^m & 2x^mz^m & 2x^my^m & 2x^m & 2y^m & 2z^m \end{pmatrix}.$$

The vector $k = [k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9]^T$ of coefficients k_i is the solution of the system $S \cdot k = P$, where $P = [-1, -1, -1, -1, -1, -1, -1, -1, -1]^T$. The number $m \geq 9$ of measured points might be large, therefore we have more equations (4) then unknowns k_i , so we minimize the function:

$$\Gamma = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} s_{i,j} k_j + 1 \right]^2, \qquad S = \left\| s_{i,j} \right\|.$$

The minimum of Γ is achieved in the zero of the derivate of Γ along all coefficients k_i , i.e. by solving:

$$\Gamma = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} s_{i,j} k_j + 1 \right]^2, \qquad l = 1, .., 9. \tag{5}$$

so k is solution of $Ak = B$, where $A = S^T S$ and $B = S^T P$. This system is solved then by the method of singular decomposition in order to minimize the computing error. In fact, $k = VW^{-1}U^T B$ where U, V are orthogonal matrices such that $U^T AV = D$, and D is a diagonal matrix:

$$D = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix W is also diagonal and it has on the diagonal as elements the eigenvalues of the matrix $A^T A$. The singular decomposition of the symmetric matrix A is given by

$$A = UDV^T.$$

The approximate equation of the surface is not sufficient for determination of the type of the surface. For this we need to find eigenvalues and rank of matrices directly from measured points. Therefore, the next step is to compute the eigenvectors of matrices
M, N. This goal is done using the Jacobi method [1].

This method is iterative, and it reduces the starting matrix $A = \|a_{p,q}\|$ to the matrix in the diagonal form by the sequence of unitary transformations $T_{p,q} = T_{p,q}(\varphi)$, φ is the angle of an appropriate chosen rotation. In each step the non-diagonal element $a_{p,q}$ is annulated, i.e. $a'_{p,q} = 0$ and $a'_{q,p} = 0$ as well by the symmetry of A. As transformations are unitary, they do not change metric properties of the surface. If $A' = T_{p,q}^T A T_{p,q}$ is the iteration step, $A' = \|a'_{p,q}\|$, the angle φ is obtained from:

$$\frac{1}{2}(a_{qq} - a_{pp})\sin 2\phi + a_{pq}\cos 2\phi = 0$$

i.e. by computing θ from:

$$\theta = \cot 2\phi \frac{\cos^2\phi - \sin^2\phi}{2\sin\phi\cos\phi} = \frac{a_{qq} - a_{pp}}{2a_{pq}}. \tag{6}$$

Therefore, for $t = \tan\theta = \frac{\sin\theta}{\cos\theta}, \theta = (1-t^2)/2t$, and we take the smaller root $\bar{t}$ of the equation $t^2 + 2t\theta - 1 = 0$. The reason is that to this smaller root $\bar{t}$ value, the rotation angle φ is smaller than 45°, and rotation appears to be more stable. The value t is given by:

$$t = \frac{sign(\theta)}{\sqrt{\theta^2 + 1} + |\theta|}. \tag{7}$$

Finding the final, diagonal form $\overline{A}$ of the matrix A, we read the eigenvalues on the diagonal of $\overline{A}$, while the number of non-zero elements on the diagonal gives the rank(A). So computed elements are sufficient to determine the type of the surface according to the displayed table.

A few words about the convergence and program implementation of the algorithm. The convergence is achieved after at most $5n^2$ rotations. Each rotation consists of $4n$ arithmetical operations, therefore the complexity of the procedure is $20n^3$. The algorithm is first implemented in Matlab. However, it appeared that this package is to slow, so the procedure is implemented again in C++ using OpenGL, the library for graphical presentation. We tested the program on various examples, most of them consisting of several thousands measured points. The obtained results were at least comparable, or better than implementations of other authors.

In astronomical applications we tested procedure for finding distribution characteristics of observations of asteroids Ceres and Pallas. So derived results are quite agreeable with results obtained with other methods.

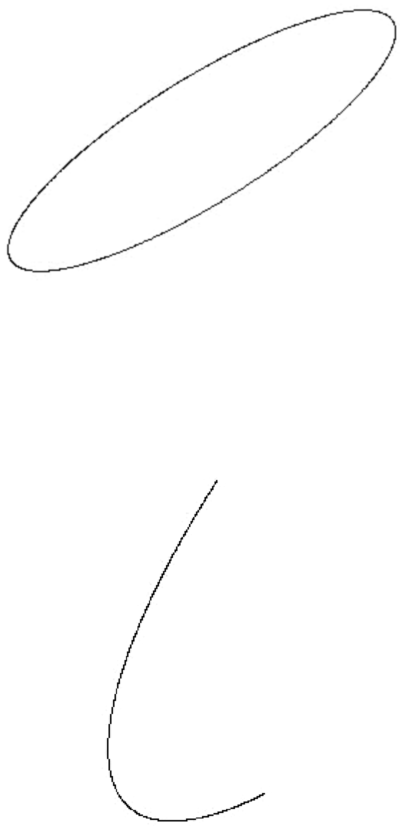

Fig 1. Ellipse and Parabola.

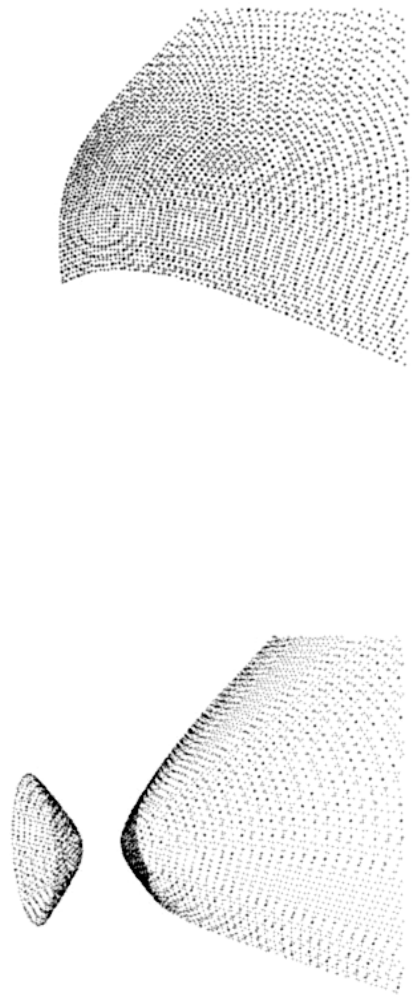

Fig 2. Paraboloid and Hyperboloid

In order to analyze performance of our method, sets of observational data of two largest asteroids in main asteroid belt were downloaded from the public database AstDys (http://hamilton.dm.unipi.it/astdys). We choose as test bodies Ceres and Pallas because of large number of their observations. As a consequence, their orbits could be determined with high accuracy. We made several tests with different number of used observations. In almost all cases our method has shown that observations concentrate around an elipse. Our method should be tested on other asteroids, with smaller number of observations. We expact to determine what are the most suitable bodies for appllying our method. Our method could be used in preliminary investigations of distribution characteristics of asteroid observations.

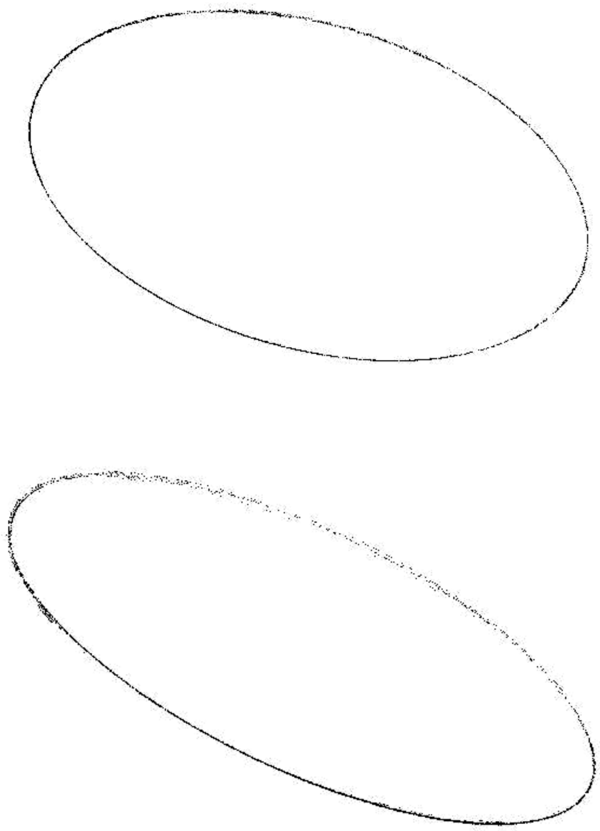

Fig 3. Ceres and Pallas.

REFERENCES

1. William H. Press, Saul A. Teukolsky, William T. Vettering, and Brian P Flannery, *Numerical Recipes in C++: The Art of Scientific Computing*, second edition, Cambridge University Press, 2002.