Radiative transfer in atmospheres of hot stars

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Stellar atmosphere

- part connecting dense stellar core and transparent interstellar medium
- “boundary layer”
- the only part of the star we directly see
- light carries the only information about astronomical objects
- light influences the state of the stellar atmosphere
  - change of ionization stages
  - change of the population numbers
  - energy transfer $\Rightarrow$ heating
  - momentum transfer $\Rightarrow$ stellar wind

Model stellar atmosphere

basic question:

- we have a spectrum -> what are the parameters of the star?
  - direct answer difficult

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  - direct answer difficult
- we solve reverse problem:
  for given stellar parameteres we determine synthetic (theoretical) spectrum

Model stellar atmosphere

standard task of stellar atmosphere physics:

- determination of space distribution of basic physical quantities – $T(\vec{r})$, $n_e(\vec{r})$, $\rho(\vec{r})$, $\vec{v}(\vec{r})$, $J_\nu(\vec{r})$, $n_i(\vec{r})$, $\ldots$
- by solving equations
  - energy equilibrium (T)
  - radiative transfer (J_ν)
  - statistical equilibrium (n_i)
  - state equation (n_e)
  - continuity (ρ)
  - motion ($\vec{v}$)

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  - continuity (ρ)
  - motion ($\vec{v}$)
- huge system of equations, approximations necessary
- once the atmospheric structure is known, detailed $I_{\mu\nu}$ can be calculated

Model stellar atmosphere

final goal – comparison with observations

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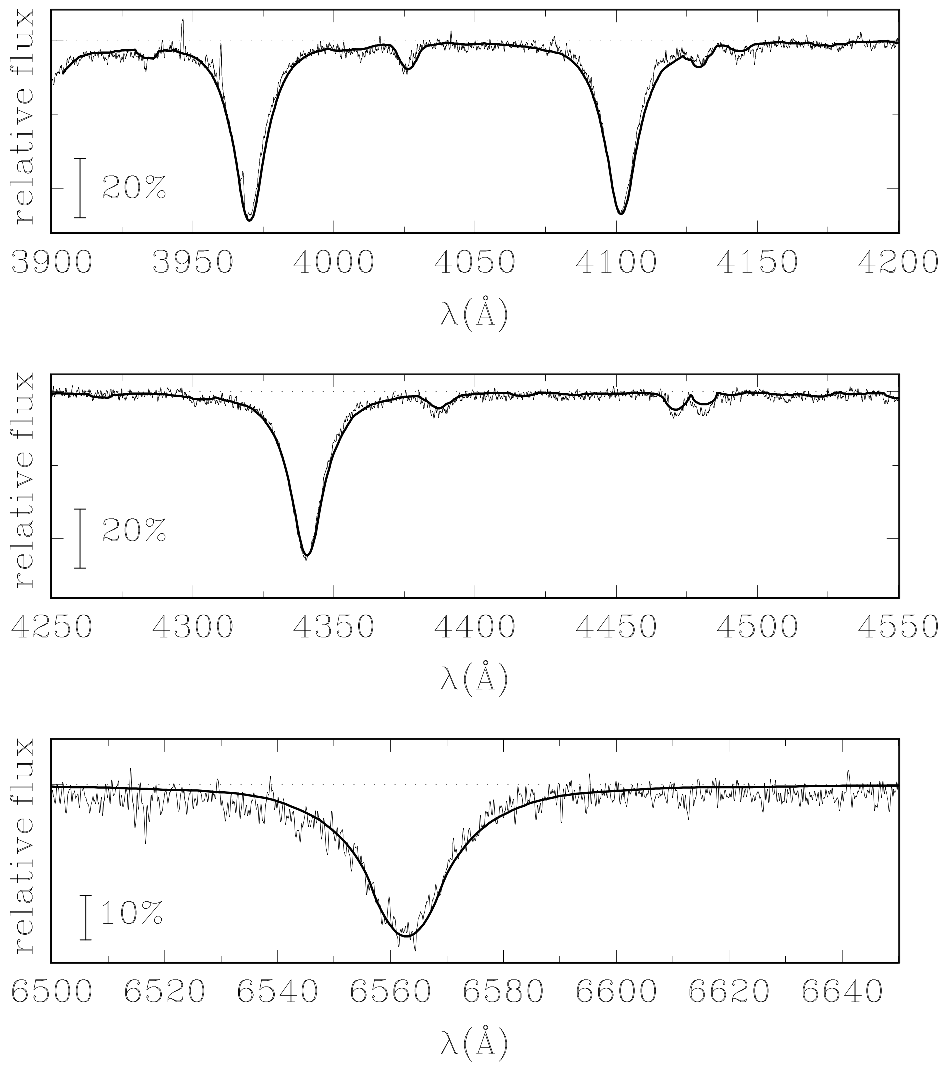
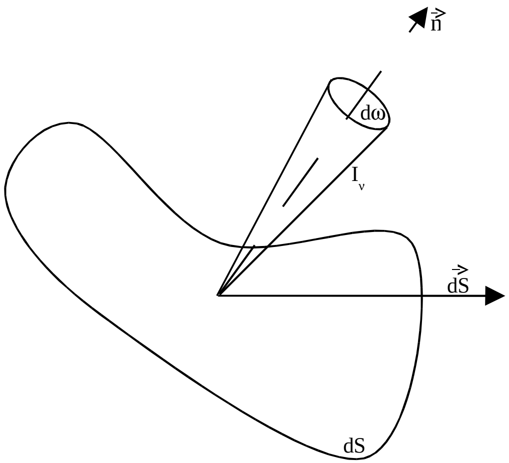


Description of radiation

- corpuscular (photons)
- electromagnetic waves
- macroscopic (phenomenological)

Specific intensity of radiation



$\delta\mathcal{E}$ – amount of transferred energy by radiation with frequencies $\langle \nu ; \nu + \mathrm{d}\nu \rangle$ through surface element $\mathrm{d}S$ to space angle $\mathrm{d}\omega$ in a time interval $\mathrm{d}t$

$$\delta\mathcal{E} = I(\vec{r}, \vec{n}, \nu, t)\,\mathrm{d}S\,\cos\theta\;\mathrm{d}\omega\;\mathrm{d}\nu\;\mathrm{d}t$$

dimension $[I] = \mathrm{erg} \cdot \mathrm{cm}^{-2} \cdot \mathrm{s}^{-1} \cdot \mathrm{Hz}^{-1} \cdot \mathrm{sr}^{-1}$

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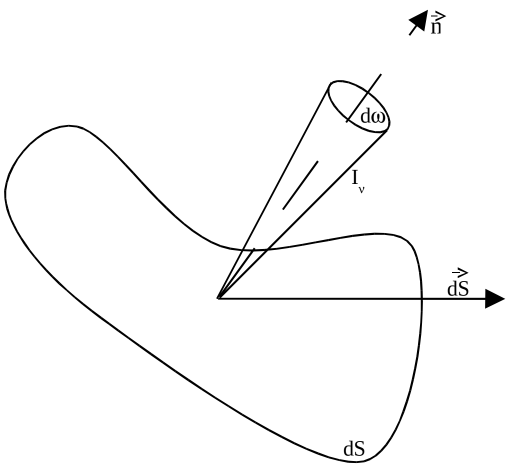

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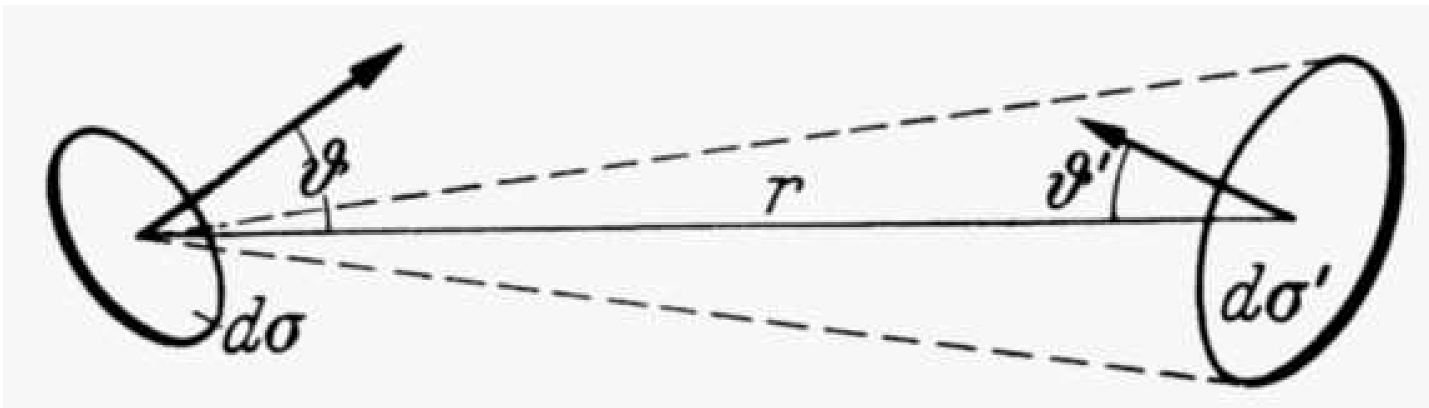

$\delta\mathcal{E} = I\,\mathrm{d}\sigma\ \cos\vartheta\ \mathrm{d}\omega\ \mathrm{d}\nu\ \mathrm{d}t$, $\delta\mathcal{E}' = I'\,\mathrm{d}\sigma'\ \cos\vartheta'\ \mathrm{d}\omega'\ \mathrm{d}\nu\ \mathrm{d}t$,

$\mathrm{d}\omega$ is a space angle, under which $\mathrm{d}\sigma'$ is seen from $\vec{r}$, $\mathrm{d}\omega = \mathrm{d}\sigma'\cos\theta'/r^2$;

similarly $\mathrm{d}\omega' = \mathrm{d}\sigma\cos\theta/r^2$;

energy conservation $\Rightarrow \delta\mathcal{E} = \delta\mathcal{E}'$, then necessarily also $I = I'$.

Mean intensity

$$J(\vec{r},\nu,t) = \frac{1}{4\pi}\oint I(\vec{r},\vec{n},\nu,t)\,\mathrm{d}\omega$$

$\mathrm{d}\omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$

dimension $[J] = \mathrm{erg}\cdot\mathrm{cm}^{-2}\cdot\mathrm{s}^{-1}\cdot\mathrm{Hz}^{-1}$

monochromatic energy density

$$E_R(\vec{r},\nu,t) = \frac{1}{c}\oint I(\vec{r},\vec{n},\nu,t)\,\mathrm{d}\omega = \frac{4\pi}{c}J(\vec{r},\nu,t)\,.$$

total energy density

$$E_R(\vec{r},t) = \frac{4\pi}{c}\int J(\vec{r},\nu,t)\,\mathrm{d}\nu = \frac{4\pi}{c}J(\vec{r},t)$$

Flux

$$\mathcal{F}(\vec{r},\nu,t) = \oint I(\vec{r},\vec{n},\nu,t)\vec{n}\,\mathrm{d}\omega$$

$\mathcal{F}\cdot\,\mathrm{d}\vec{S}$ – net energy flux across $\mathrm{d}S$ (arbitrarily orientated)
dimension $[\mathcal{F}] = \mathrm{erg}\cdot\mathrm{cm}^{-2}\cdot\mathrm{s}^{-1}\cdot\mathrm{Hz}^{-1}$

Radiation pressure tensor

second moment of intensity

$$\mathsf{P}(\vec{r},\nu,t) = \frac{1}{c}\oint I(\vec{r},\vec{n},\nu,t)\vec{n}\vec{n}\,\mathrm{d}\omega$$

dimension $[\mathsf{P}] = \mathrm{erg}\cdot\mathrm{cm}^{-3}\cdot\mathrm{Hz}^{-1}$; symmetric tensor ($P_{ij} = P_{ji}$).

physical meaning of P:

$$P_{ij}(\vec{r},\nu,t) = \oint \left[f_R(\vec{r},\vec{n},\nu,t)cn_i\right] \times \frac{h\nu n_j}{c}\,\mathrm{d}\omega,$$

it is the flux of the momentum in the direction n_j caused by radiation with a frequency ν across surface element oriented perpendicular to n_i, which corresponds to a definition of a pressure in an arbitrary continuum (from fluid mechanics)

Radiation pressure tensor

mean radiation pressure $\bar{P} = \frac{1}{3}\mathrm{Tr}\mathsf{P}$, since $n_x^2 + n_y^2 + n_z^2 = 1$, we obtain

$$\bar{P}(\vec{r}, \nu, t) = \frac{1}{3c} \oint I(\vec{r}, \vec{n}, \nu, t)\, \mathrm{d}\omega = \frac{1}{3} E_R(\vec{r}, \nu, t)$$

valid for isotropic radiation, for anisotropic is the ratio $\bar{P}/E_R > 1/3$. The ratio

$$\mathsf{f}(\vec{r}, \nu, t) = \mathsf{P}(\vec{r}, \nu, t)/E_R(\vec{r}, \nu, t)$$

is the Eddington tensor.

Equilibrium distribution

photons – bosons (from statistical physics)
eqilibrium occupation number

$$N_\alpha = \frac{1}{e^{\frac{\varepsilon - \mu}{kT}} - 1}$$

for photons: $\varepsilon = h\nu$, $\mu = 0$ and using $I(\vec{n}, \nu) = \frac{2h\nu^3}{c^2} N_\alpha$

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$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

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$$B_\nu = I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$ Planck function

Equilibrium distribution

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limiting cases

$$\frac{h\nu}{kT} \gg 1 \quad B_\nu \approx \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}$$ Wien

$$\frac{h\nu}{kT} \ll 1 \quad B_\nu \approx \frac{2\nu^2 kT}{c^2}$$ Rayleigh-Jeans

Equilibrium distribution

Integrated Planck function over frequencies

$$B(T) = \int_0^\infty B_\nu(T)\,\mathrm{d}\nu = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\,\mathrm{d}x = \frac{\sigma}{\pi}T^4$$

where $\sigma = \dfrac{2\pi^5 k^4}{15c^2h^3} = 5.67 \cdot 10^{-5}\mathrm{erg\,cm^{-2}s^{-2}K^{-4}}$

is the Stefan-Boltzmann constant

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total energy density

$$E_R^* = \frac{4\pi}{c}B(T) = \frac{4\sigma}{c}T^4 = a_R T^4 \qquad (2)$$

Stephan's law ($a_R = 7.56 \cdot 10^{-15}\mathrm{erg\,cm^{-2}K^{-4}}$)

Interaction of matter and radiation

basic types of interaction

- absorption / emission
- scattering

we characterize them using macroscopic quantities

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difference between true absorption and scattering

Interaction of matter and radiation

examples of scattering processes

- atom excitation followed by de-excitation (absorption and emission in the same line)
- scattering on free electrons (Thomson or Compton)

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examples of true absorption processes

- photon ionizes atom
- photon is absorbed by an electron moving in the atom field (free-free absorption); reverse process is known as bremsstrahlung
- atom is photoexcited without subsequent emission (photon is collisionally destroyed – thermalized)

Interaction of matter and radiation

examples of unclear processes

- three-level atom – different sequences of processes
- solution of the equations of statistical equilibrium

Interaction of matter and radiation

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other types of interactions

- acceleration of charged particles in a Coulombic fiels of another charged particle (bremsstrahlung)
- radiation of moving charged particles
  - cyclotron (non-relativistic)
  - synchrotron (relativistic)

Absorption coefficient

extinction coefficients, opacity, total absorption coefficient

$$\delta E = \chi(\vec{r},\vec{n},\nu,t) I(\vec{r},\vec{n},\nu,t)\,\mathrm{d}S\,\mathrm{d}s\,\mathrm{d}\omega\,\mathrm{d}\nu\,\mathrm{d}t$$

dimension $[\chi] = \mathrm{cm}^{-1}$

$\dfrac{1}{\chi}$ – mean free photon path

static medium $\Rightarrow$ χ isotropic

division of the total absorption coefficient to true absorption κ and scattering σ

$$\chi(\vec{r},\vec{n},\nu,t) = \kappa(\vec{r},\vec{n},\nu,t) + \sigma(\vec{r},\vec{n},\nu,t)$$

Emission coefficient

emission coefficient, emissivity

$$\delta E = \eta(\vec{r}, \vec{n}, \nu, t)\, \mathrm{d}S\, \mathrm{d}s\, \mathrm{d}\omega\, \mathrm{d}\nu\, \mathrm{d}t$$

dimension $[\eta] = \mathrm{erg} \cdot \mathrm{cm}^{-3} \cdot \mathrm{sr}^{-1} \cdot \mathrm{s}^{-1} \cdot \mathrm{Hz}^{-1}$

division of the total emission coefficient to thermal emission η^{th} and scattering η^{S}

$$\eta(\vec{r}, \vec{n}, \nu, t) = \eta^{\mathrm{th}}(\vec{r}, \vec{n}, \nu, t) + \eta^{S}(\vec{r}, \vec{n}, \nu, t)$$

Spontaneous and stimulated emission

$$\eta = \eta^{\rm spont} + \eta^{\rm stim}$$

$\eta^{\rm spont}$ spontaneous emission (independent of radiation, isotropic)

$\eta^{\rm stim}$ stimulated emission – proportional to the radiation field

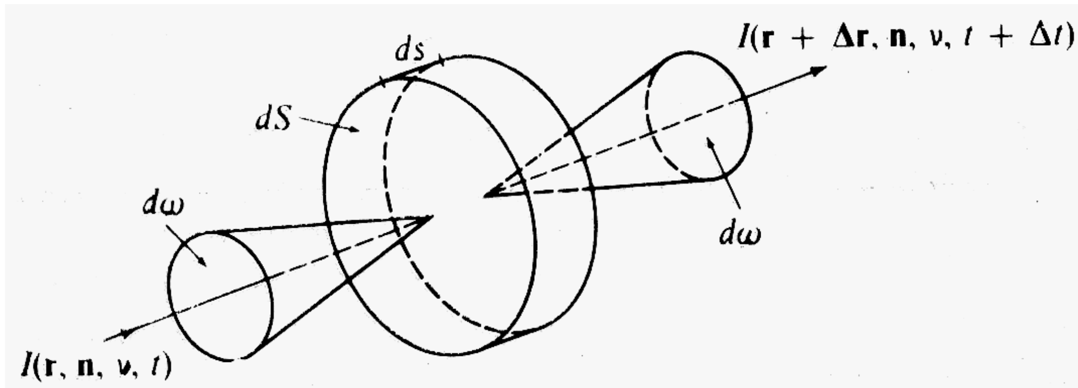
the emitted energy during stimulated emission (from quantum mechanics)

$$\eta^{\rm stim}(\vec{r},\vec{n},\nu,t) = \frac{c^2}{2h\nu^3}\eta^{\rm spont}(\vec{r},\vec{n},\nu,t)I(\vec{r},\vec{n},\nu,t)$$

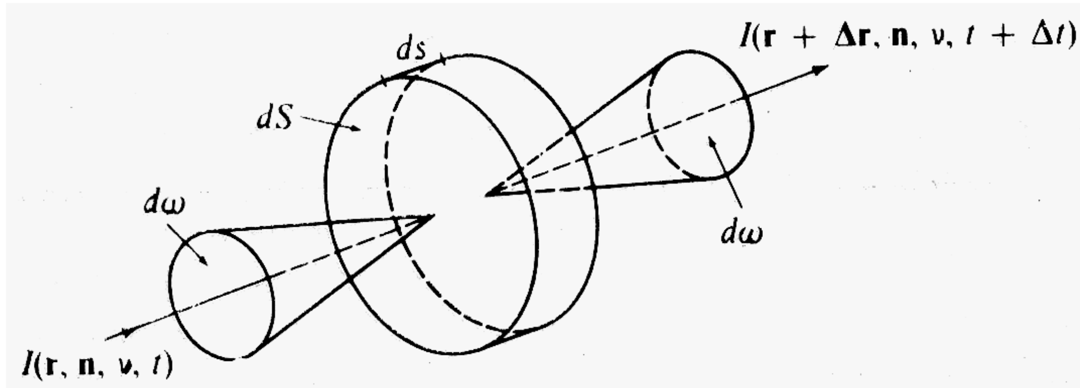
stimulated emission often treated as negative absorption

Radiative transfer equation

basic equation of stellar atmospheres



Radiative transfer equation



$$\left[I\left(\vec{r}+\Delta\vec{r},\vec{n},\nu,t+\Delta t\right)-I(\vec{r},\vec{n},\nu,t)\right]\;\mathrm{d}S\;\mathrm{d}\omega\;\mathrm{d}\nu\;\mathrm{d}t=$$

$$=\left[\eta(\vec{r},\vec{n},\nu,t)-\chi(\vec{r},\vec{n},\nu,t)I(\vec{r},\vec{n},\nu,t)\right]\;\mathrm{d}s\;\mathrm{d}S\;\mathrm{d}\omega\;\mathrm{d}\nu\;\mathrm{d}t$$

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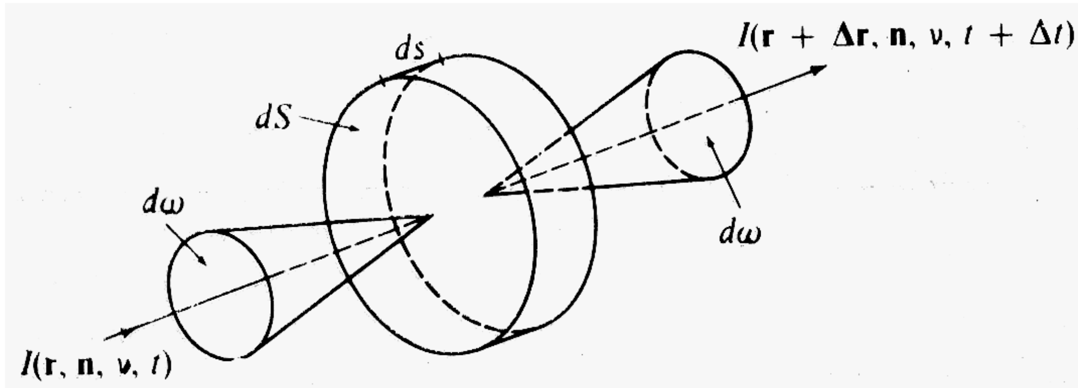

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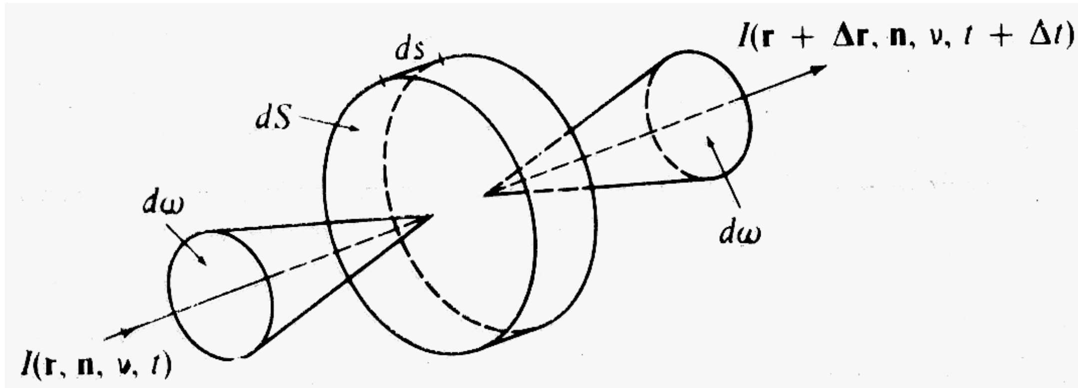
$$= \left[\eta(\vec{r},\vec{n},\nu,t)-\chi(\vec{r},\vec{n},\nu,t)I(\vec{r},\vec{n},\nu,t)\right]\ \mathrm{d}s\ \mathrm{d}S\ \mathrm{d}\omega\ \mathrm{d}\nu\ \mathrm{d}t$$

$$I\left(\vec{r}+\Delta\vec{r},\vec{n},\nu,t+\Delta t\right)=I(\vec{r},\vec{n},\nu,t)+\left[\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right]I(\vec{r},\vec{n},\nu,t)\ \mathrm{d}s$$

Radiative transfer equation

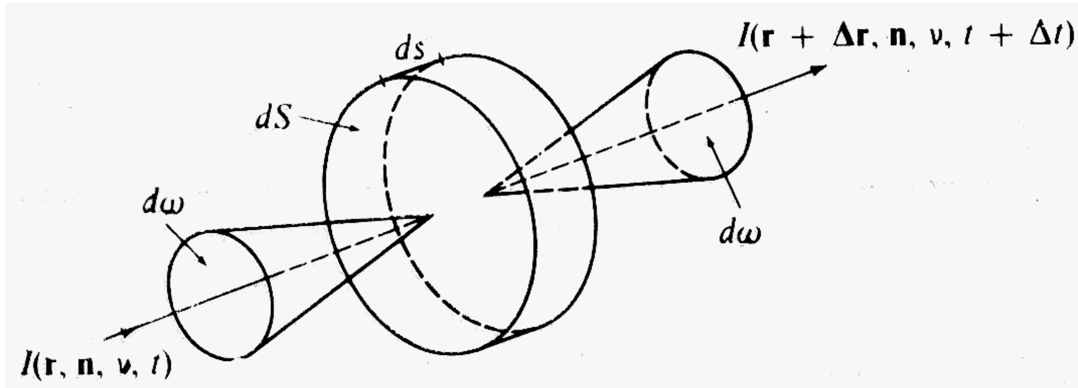


$$[I(\vec{r}+\Delta\vec{r},\vec{n},\nu,t+\Delta t) - I(\vec{r},\vec{n},\nu,t)]\ \mathrm{d}S\ \mathrm{d}\omega\ \mathrm{d}\nu\ \mathrm{d}t =$$

$$= [\eta(\vec{r},\vec{n},\nu,t) - \chi(\vec{r},\vec{n},\nu,t)I(\vec{r},\vec{n},\nu,t)]\ \mathrm{d}s\ \mathrm{d}S\ \mathrm{d}\omega\ \mathrm{d}\nu\ \mathrm{d}t$$

$$I(\vec{r}+\Delta\vec{r},\vec{n},\nu,t+\Delta t) - I(\vec{r},\vec{n},\nu,t) = \left[\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right] I(\vec{r},\vec{n},\nu,t)\ \mathrm{d}s$$

Radiative transfer equation



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$$= \left[\eta(\vec{r},\vec{n},\nu,t)-\chi(\vec{r},\vec{n},\nu,t)I(\vec{r},\vec{n},\nu,t)\right]\ \mathrm{d}s\ \mathrm{d}S\ \mathrm{d}\omega\ \mathrm{d}\nu\ \mathrm{d}t$$

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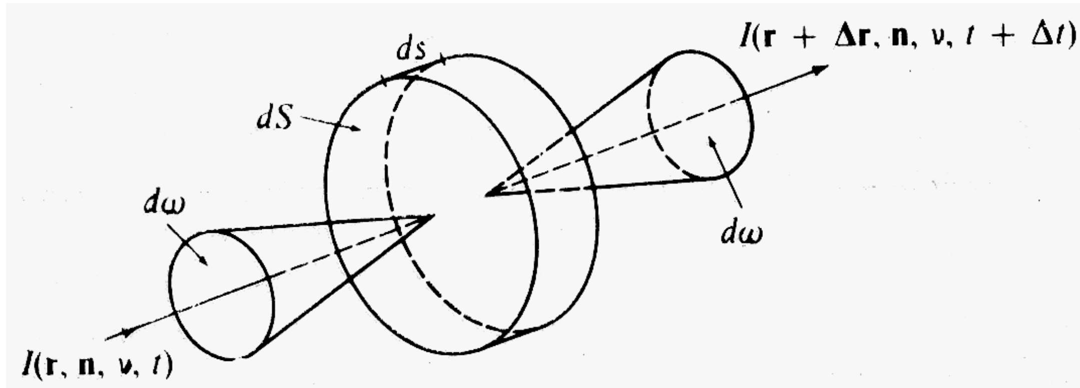
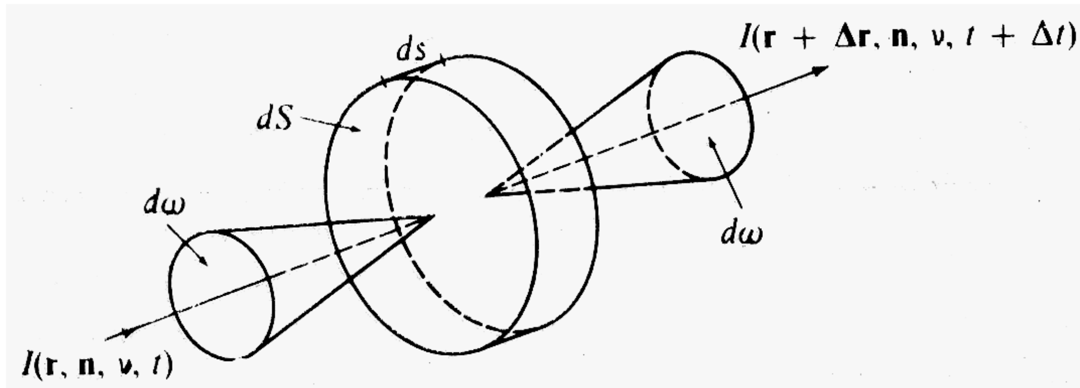


$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right] I(\vec{r},\vec{n},\nu,t) =$$

$$= \left[\eta(\vec{r},\vec{n},\nu,t) - \chi(\vec{r},\vec{n},\nu,t) I(\vec{r},\vec{n},\nu,t)\right]$$

Radiative transfer equation



$$\left[\frac{1}{c}\frac{\partial}{\partial t} + (\vec{n}\cdot\nabla)\right] I(\vec{r},\vec{n},\nu,t) =$$

$$= [\eta(\vec{r},\vec{n},\nu,t) - \chi(\vec{r},\vec{n},\nu,t) I(\vec{r},\vec{n},\nu,t)]$$

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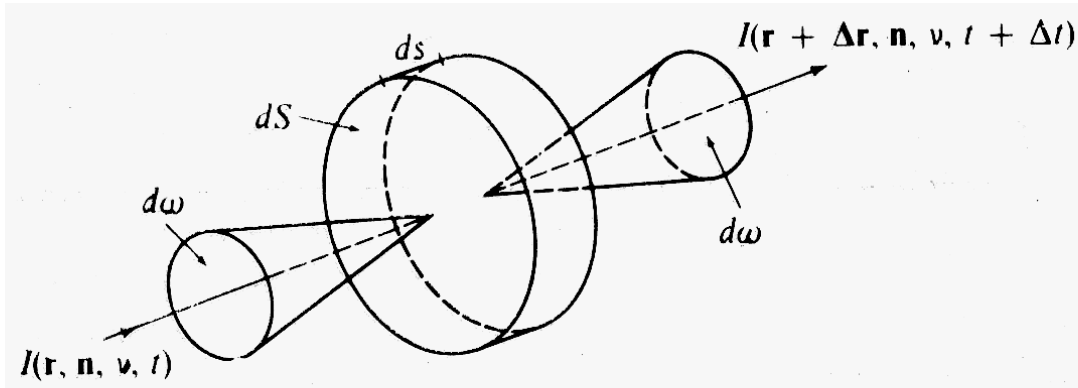


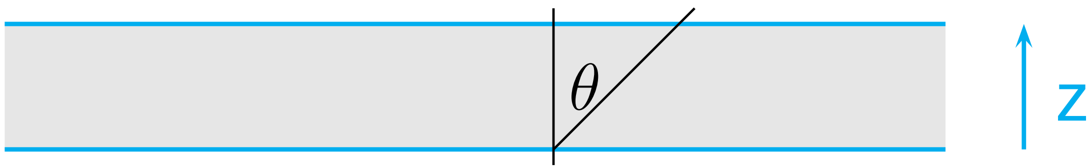
$$\left[\frac{1}{c}\,\frac{\partial}{\partial t} + (\vec{n}\cdot\nabla)\right] I(\vec{r},\vec{n},\nu,t) =$$

$$= \left[\eta(\vec{r},\vec{n},\nu,t) - \chi(\vec{r},\vec{n},\nu,t) I(\vec{r},\vec{n},\nu,t)\right]$$

it is an integrodifferential equation

Planar geometry

1-dimensional plane-parallel atmosphere



$$n_z = \frac{\mathrm{d}z}{\mathrm{d}s} = \cos\theta = \mu \qquad \frac{\partial}{\partial x} = 0 \quad \frac{\partial}{\partial y} = 0$$

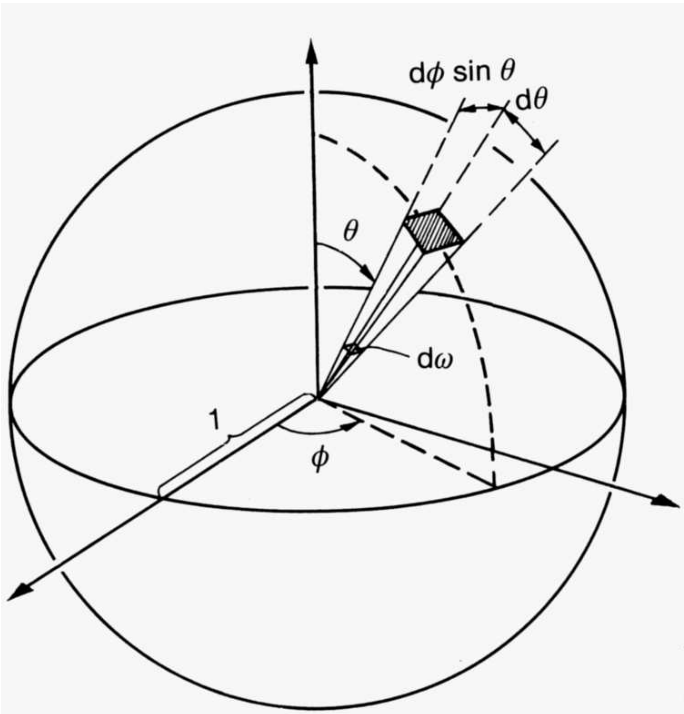
$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial z}\right] I(z,\mu,\nu,t) = \eta(z,\mu,\nu,t) - \chi(z,\mu,\nu,t) I(z,\mu,\nu,t)$$

static case:

$$\mu\,\frac{\partial I(z,\mu,\nu)}{\partial z} = \eta(z,\mu,\nu) - \chi(z,\mu,\nu) I(z,\mu,\nu)$$

Spherically symmetric geometry

one-dimensional spherically symmetric atmosphere



$$\mathrm{d}r = \cos\theta \ \mathrm{d}s, \ r\,\mathrm{d}\theta = -\sin\theta \ \mathrm{d}s$$

$$\frac{\partial}{\partial s} \rightarrow \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} = \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu}$$

Spherically symmetric geometry

$$\frac{\partial}{\partial s} \rightarrow \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} = \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}$$

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right] I(r,\mu,\nu,t) = \eta(r,\mu,\nu,t) - \chi(r,\mu,\nu,t) I(r,\mu,\nu,t)$$

static case:

$$\mu\frac{\partial I(r,\mu,\nu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(r,\mu,\nu)}{\partial\mu} = \eta(r,\mu,\nu) - \chi(r,\mu,\nu) I(r,\mu,\nu)$$

Optical depth

for the direction s we introduce

$$\mathrm{d}\tau(s,\nu) \equiv -\chi(s,\nu)\ \mathrm{d}s$$

$$\tau(s,\nu) = \int_{s}^{s_{\max}} \chi(s',\nu)\ \mathrm{d}s'$$

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$$\tau(s,\nu) = \int_{s}^{s_{\max}} \chi(s',\nu)\ \mathrm{d}s'$$

$1/\chi$ – mean free photon path $\Rightarrow$
τ – number of mean free photon paths between s and s'

optical depth is the distance in units of the mean free path

Source function

$$S(\vec{r},\vec{n},\nu) = \frac{\eta(\vec{r},\vec{n},\nu)}{\chi(\vec{r},\vec{n},\nu)}$$

radiative transfer equation along a ray s with the optical depth ($\tau_{s\nu} \equiv \tau(s,\nu)$)

$$\frac{\partial I_{\nu\mu}}{\partial \tau_{s\nu}} = I_{\nu\mu} - S_{\nu\mu}$$

the source function is proportional to number of photons emitted per unit optical depth.

Boundary conditions of the RTE

Semi-infinite medium —
example: stellar atmosphere of isolated stars

upper boundary condition

$$I(\tau_\nu = 0, \mu, \nu) = 0$$

lower boundary condition

$$\lim_{\tau_\nu \to \infty} I(\tau_\nu, \mu, \nu) e^{-\tau_\nu/\mu} = 0$$

often expressed by diffusion approximation – later

Boundary conditions of the RTE

Finite slab, spherical shell of total optical depth T_ν
example: prominences, planetary nebulae, circumstelar shell;
also stellar atmosphere

upper boundary condition

$$I(\tau_\nu = 0, \mu, \nu) = I^-(\mu, \nu) \qquad (-1 \leq \mu \leq 0)$$

lower boundary condition

$$I(\tau_\nu = T_\nu, \mu, \nu) = I^+(\mu, \nu) \qquad (0 \leq \mu \leq 1)$$

Boundary conditions of the RTE

Symmetric layer —
example: accretion disk

upper boundary condition

$$I(\tau_\nu = 0, \mu, \nu) = I^-(\mu, \nu) \qquad (-1 \le \mu \le 0)$$

lower boundary condition

$$I(\tau_\nu = T_\nu, \mu, \nu) = I(\tau_\nu = T_\nu, -\mu, \nu)$$

Moments of the RTE

we drop explicit dependence on t and $\vec{r}$

$$\frac{1}{c}\,\frac{\partial I_\nu(\vec{n})}{\partial t} + (\vec{n}\cdot\nabla)\,I_\nu(\vec{n}) = [\eta_\nu(\vec{n}) - \chi_\nu(\vec{n})I_\nu(\vec{n})]$$

integrating over $\omega \longrightarrow$ energy equation

$$\frac{\partial E_\nu}{\partial t} + \nabla\cdot\mathcal{F}_\nu = \oint [\eta_\nu(\vec{n}) - \chi_\nu(\vec{n})I_\nu(\vec{n})]\;\mathrm{d}\omega$$

multiplying by $\vec{n}$ and integrating over $\omega \longrightarrow$ radiation momentum equation

$$\frac{1}{c^2}\,\frac{\partial \vec{\mathcal{F}}_\nu}{\partial t} + \nabla\cdot\mathsf{P}_\nu = \frac{1}{c}\oint \vec{n}\,[\eta_\nu(\vec{n}) - \chi_\nu(\vec{n})I_\nu(\vec{n})]\;\mathrm{d}\omega$$

$\vec{G}_R = \mathcal{F}/c^2$ – radiation momentum density

Moments of the RTE

for isotropic χ and η

$$\frac{\partial E_\nu}{\partial t} + \nabla \cdot \vec{\mathcal{F}}_\nu = \eta_\nu - \chi_\nu J_\nu$$

$$\frac{\partial \vec{G}_\nu}{\partial t} + \nabla \cdot \mathsf{P}_\nu = -\frac{1}{c}\chi_\nu \vec{\mathcal{F}}_\nu$$

For moments of the specific intensity holds

$$\begin{pmatrix} J_\nu \\ \vec{H}_\nu \\ \mathsf{K}_\nu \end{pmatrix} = \frac{1}{4\pi}\begin{pmatrix} cE_\nu \\ \vec{\mathcal{F}}_\nu \\ c\mathsf{P}_\nu \end{pmatrix} = \frac{1}{4\pi}\oint \begin{pmatrix} 1 \\ \vec{n} \\ \vec{n}\vec{n} \end{pmatrix} I_\nu \, \mathrm{d}\omega$$

Solution of the RTE

No absorption, no emission: $\chi = 0$, $\eta = 0$

$$\frac{\mathrm{d}I}{\mathrm{d}z} = 0 \qquad \Rightarrow \qquad I = \mathrm{const}$$

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Only emission: $\chi = 0$, $\eta > 0$, optically thin medium (planetary nebulae)

$$\mu \frac{\mathrm{d}I}{\mathrm{d}z} = \eta \qquad \Rightarrow \qquad I(z,\mu) = I(0,\mu) + \int_0^z \eta(z') \frac{\mathrm{d}z'}{\mu}$$

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Only absorption: $\chi > 0$, $\eta = 0$, absorption in the Earth's atmosphere

$$\mu \frac{\mathrm{d}I}{\mathrm{d}\tau} = I \qquad \Rightarrow \qquad I(0,\mu) = I(\tau,\mu) e^{-\frac{\tau}{\mu}}$$

Solution of the RTE

Absorption and emission:

$$\mu \frac{\mathrm{d}I}{\mathrm{d}\tau} = I - S$$

Solution of the RTE

Absorption and emission:

$$\mu \, \frac{\mathrm{d}I}{\mathrm{d}\tau} \mathrm{e}^{-\frac{\tau}{\mu}} = I \mathrm{e}^{-\frac{\tau}{\mu}} - S \mathrm{e}^{-\frac{\tau}{\mu}}$$

integration factor $\mathrm{e}^{-\frac{\tau}{\mu}}$

Solution of the RTE

Absorption and emission:

$$\mu \frac{\mathrm{d}I}{\mathrm{d}\tau}\mathrm{e}^{-\frac{\tau}{\mu}} = I\mathrm{e}^{-\frac{\tau}{\mu}} - S\mathrm{e}^{-\frac{\tau}{\mu}}$$

integration factor $\mathrm{e}^{-\frac{\tau}{\mu}}$

$$\frac{\mathrm{d}\left(I\mathrm{e}^{-\frac{\tau}{\mu}}\right)}{\mathrm{d}\tau} = -\frac{S}{\mu}\,\mathrm{e}^{-\frac{\tau}{\mu}}$$

Solution of the RTE

Absorption and emission:

$$\frac{\mathrm{d}\left(I\mathrm{e}^{-\frac{\tau}{\mu}}\right)}{\mathrm{d}\tau} = -\frac{S}{\mu}\,\mathrm{e}^{-\frac{\tau}{\mu}}$$

By integration we obtain

$$I(\tau_1,\mu) = I(\tau_2,\mu)e^{-\frac{\tau_2-\tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S(t)e^{-\frac{t-\tau_1}{\mu}}\,\frac{\mathrm{d}t}{\mu}$$

Solution of the RTE

Absorption and emission:

$$\frac{\mathrm{d}\left(I\mathrm{e}^{-\frac{\tau}{\mu}}\right)}{\mathrm{d}\tau} = -\frac{S}{\mu}\,\mathrm{e}^{-\frac{\tau}{\mu}}$$

By integration we obtain

$$I(\tau_1, \mu) = I(\tau_2, \mu)e^{-\frac{\tau_2 - \tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S(t)e^{-\frac{t-\tau_1}{\mu}}\,\frac{\mathrm{d}t}{\mu}$$

First term at the right hand side describes dilution of radiation by absorption, second term describes radiation emitted between τ_1 a τ_2

Solution of the RTE

$$I(\tau_1, \mu) = I(\tau_2, \mu)e^{-\frac{\tau_2 - \tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S(t)e^{-\frac{t-\tau_1}{\mu}} \frac{\mathrm{d}t}{\mu}$$

Semi-infinite atmosphere: ($\tau_1 = 0$, $\tau_2 \to \infty$)

$$I(0, \mu) = \int_0^\infty S(t)\, e^{-\frac{t}{\mu}} \, \frac{\mathrm{d}t}{\mu}$$

Solution of the RTE

Semi-infinite atmosphere: $(\tau_1 = 0,\ \tau_2 \to \infty)$

$$I(0,\mu) = \int_0^\infty S(t)\, e^{-\frac{t}{\mu}}\, \frac{\mathrm{d}t}{\mu}$$

Solution of the RTE

Semi-infinite atmosphere: ($\tau_1 = 0$, $\tau_2 \rightarrow \infty$)

$$I(0,\mu) = \int_0^\infty S(t)\, e^{-\frac{t}{\mu}}\, \frac{\mathrm{d}t}{\mu}$$

Semi-infinite atmosphere with linear S: ($S\left(\tau\right) = a + b\tau$)

$$I(0,\mu) = a + b\mu = S(\tau = \mu)$$ Eddington-Barbier relation

Intensity of the emergent radiation in the perpendicular direction ($\mu = 1$) equals the source function at the unit optical depth, Eddington-Barbier relation offers for many cases a suitable approximation of the emergent intensity.

Solution of the RTE

Finite homogeneous slab: $(S = \mathrm{const})$

$$\tau_1 = 0$$

$$\tau_2 = T < \infty$$

$$I(0,1) = S\left(1 - e^{-T}\right)$$

Probabilistic interpretaion

description, what happens (absorbed, emmited, scattered) with *one* photon – different from intensity (which uses ensemble of photons)
consider only absorption
transfer equation

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = I,$$

solution $(\tau) = I(0)e^{-\tau}$
probability that the photon in NOT absorbed

$$p(\tau) = e^{-\tau}$$

probability of absorption

$$p_a(\tau) = 1 - e^{-\tau}$$

for very small $\delta\tau \ll 1$

$$p_a(\delta\tau) = \delta\tau$$

photon travels distance τ and then it is absorbed in $\delta\tau$

$$p(\tau)\,\mathrm{d}\tau = e^{-\tau}\,\mathrm{d}\tau$$

for arbitrary direction form $\tau = 0$

$$\overline{p}(\tau)\,\mathrm{d}\tau = \int_0^1 e^{-\frac{\tau}{\mu}}\,\frac{\mathrm{d}\tau}{\mu}\,\mathrm{d}\mu = E_1(\tau)\,\mathrm{d}\tau$$

Diffusion approximation

$p_a(\tau) \approx 1$, $S_\nu \to B_\nu$
Taylor expansion S_ν for $t_\nu \geq \tau_\nu$

$$S_\nu(t_\nu) = \sum_{n=0}^{\infty} \frac{\mathrm{d}^n B}{\mathrm{d}\tau_\nu^n} \frac{(t_\nu - \tau_\nu)^n}{n!}$$

substituting to the formal solution
$I_\nu(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu(t) e^{-\frac{t-\tau_\nu}{\mu}} \frac{\mathrm{d}t}{\mu}$

$$I_\nu(\tau_\nu, \mu) = \sum_{n=0}^{\infty} \mu^n \frac{\mathrm{d}^n B}{\mathrm{d}\tau_\nu^n} = B_\nu(\tau_\nu) + \mu \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} + \mu^2 \frac{\mathrm{d}^2 B_\nu}{\mathrm{d}\tau_\nu^2} + \ldots$$

Diffusion approximation

moments

$$J_\nu = \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{\mathrm{d}^{2n} B}{\mathrm{d}\tau_\nu^{2n}} = B_\nu(\tau_\nu) + \frac{1}{3} \frac{\mathrm{d}^2 B_\nu}{\mathrm{d}\tau_\nu^2} + \dots$$

$$H_\nu = \sum_{n=0}^{\infty} \frac{1}{2n+3} \frac{\mathrm{d}^{2n+1} B}{\mathrm{d}\tau_\nu^{2n+1}} = \frac{1}{3} \frac{\mathrm{d} B_\nu}{\mathrm{d}\tau_\nu} + \dots$$

$$K_\nu = \sum_{n=0}^{\infty} \frac{1}{2n+3} \frac{\mathrm{d}^{2n} B}{\mathrm{d}\tau_\nu^{2n}} = \frac{1}{3} B_\nu(\tau_\nu) + \frac{1}{5} \frac{\mathrm{d}^2 B_\nu}{\mathrm{d}\tau_\nu^2} + \dots$$

Diffusion approximation

large depths

$$I_\nu(\tau_\nu, \mu) \approx B_\nu(\tau_\nu) + \mu \frac{\partial B_\nu}{\partial \tau_\nu}$$

$$J_\nu \approx B_\nu(\tau_\nu)$$

$$H_\nu \approx \frac{1}{3} \frac{\partial B_\nu}{\partial \tau_\nu}$$

$$K_\nu \approx \frac{1}{3} B_\nu(\tau_\nu)$$

$\Rightarrow$

$$f_K \approx \frac{1}{3}$$ izotropic radiation

Diffusion approximation

Total radiation flux

integration over frequencies

$$H = \int_0^\infty H_\nu \, \mathrm{d}\nu = \int_0^\infty \frac{1}{3} \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \, \mathrm{d}\nu$$

Diffusion approximation

Total radiation flux

integration over frequencies

$$H = \int_0^\infty \frac{1}{3} \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \, \mathrm{d}\nu = - \int_0^\infty \frac{1}{3} \frac{1}{\chi_\nu} \frac{\mathrm{d}B_\nu}{\mathrm{d}z}$$

Diffusion approximation

Total radiation flux

integration over frequencies

$$H = -\int_0^\infty \frac{1}{3}\frac{1}{\chi_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}z} = -\int_0^\infty \frac{1}{3}\frac{1}{\chi_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}z}\,\mathrm{d}\nu$$

Diffusion approximation

Total radiation flux

integration over frequencies

$$H = -\int_0^\infty \frac{1}{3}\frac{1}{\chi_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}z}\,\mathrm{d}\nu = -\frac{1}{3}\frac{\mathrm{d}T}{\mathrm{d}z}\int_0^\infty \frac{1}{\chi_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu$$

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Total radiation flux

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Rosseland mean opacity χ_R

$$\frac{1}{\chi_R}\frac{\mathrm{d}B}{\mathrm{d}T} = \int_0^\infty \frac{1}{\chi_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu$$

Diffusion approximation

Total radiation flux

integration over frequencies

$$H = -\frac{1}{3}\frac{\mathrm{d}T}{\mathrm{d}z}\int_0^\infty \frac{1}{\chi_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu = -\left(\frac{1}{3}\frac{1}{\chi_R}\frac{\mathrm{d}B}{\mathrm{d}T}\right)\frac{\mathrm{d}T}{\mathrm{d}z}$$

Rosseland mean opacity χ_R

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Diffusion approximation

Total radiation flux

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Rosseland mean opacity χ_R

$$\frac{1}{\chi_R}\frac{\mathrm{d}B}{\mathrm{d}T} = \int_0^\infty \frac{1}{\chi_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu$$

$\Rightarrow$ since we see star $H > 0$, temperature must grow inwards the star
Rosseland opacity gives correct temperature structure

Thermodynamic equilibrium

conditions for equilibrium

- $t_{\text{relaxation}} \ll t_{\text{macroscopic changes}}$
- $l_{\text{macroscopic changes}} \ll \bar{l}_{\text{free path}}$
- $t_{\text{relaxation}} \ll t_{\text{inelastic collisions}}$
- for $t_{\text{relaxation}} \gtrsim t_{\text{inelastic collisions}}$ colliding particles have to be in equilibrium

Hubený 1976, PhD thesis

Thermodynamic equilibrium

distributions in equilibrium

- electron (and other particle) velocities
  – *Maxwellian distribution*

$$f(v)\,\mathrm{d}v = \frac{1}{v_0\sqrt{\pi}} e^{-\frac{v^2}{v_0^2}}\,\mathrm{d}v$$

most probable speed: $v_0 = \sqrt{\frac{2kT}{m_e}}$

Thermodynamic equilibrium

distributions in equilibrium

- atomic level populations
  - *Boltzmann distribution*

$$\frac{n_i^*}{n_0^*} = \frac{g_i}{g_0} e^{-\frac{\chi_i}{kT}}$$

- ionization degrees distribution
  - *Saha equation*

$$\frac{N_j^*}{N_{j+1}^*} = n_e \frac{U_j(T)}{2U_{j+1}(T)} \left(\frac{h^2}{2\pi m_e kT} \right)^{\frac{3}{2}} e^{\frac{\chi_{Ij}}{kT}}$$

Thermodynamic equilibrium

distributions in equilibrium

- radiation field – *Planck distribution*

$$B_\nu(T) = \frac{2h\nu}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

Thermodynamic equilibrium

distributions in equilibrium

electron velocities	–	Maxwellian distribution
level populations	–	Saha-Boltzmann distribution
radiation field	–	Planck distribution

Thermodynamic equilibrium

distributions in equilibrium

electron velocities	–	Maxwellian distribution
level populations	–	Saha-Boltzmann distribution
radiation field	–	Planck distribution
		contradicts observations

Thermodynamic equilibrium

distributions in equilibrium

electron velocities	–	Maxwellian distribution
level populations	–	Saha-Boltzmann distribution
radiation field	–	~~Planck distribution~~
		contradicts observations

Local thermodynamic equilibrium

- locally equilibrium distributions
  (we ignore the dependence $T(\vec{r})$, $N(\vec{r})$)
  electron velocities – Maxwellian distribution
  level populations – Saha-Boltzmann distribution
- non-equilibrium distribution
  radiation field – calculated by RTE solution

$$\mu \frac{\mathrm{d}I_{\mu\nu}}{\mathrm{d}z} = \eta_\nu - \chi_\nu I_{\mu\nu}$$

with the source function equal to the Planck function

$$S_\nu = \frac{\eta_\nu}{\chi_\nu} = B_\nu$$

Local thermodynamic equilibrium

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Statistical equilibrium

usually called NLTE or non-LTE

- equilibrium distribution

  electron velocities – Maxwellian distribution

- non-equilibrium distributions

  level populations – statistical equilibrium

  radiation field – calculated by RTE solution

Microscopic processes

particle collisions

- elastic collisions (e–e, e–H, e–H^+, e–He, H–H, H–He, ...) maintain equuilibrium velocity distribution
- inelastic collisions with electrons
  - excitation: $\mathrm{e}(v) + \mathrm{X} \rightarrow \mathrm{e}(v' < v) + \mathrm{X}^*$
  - deexcitation: $\mathrm{e}(v) + \mathrm{X}^* \rightarrow \mathrm{e}(v' > v) + \mathrm{X}$
  - ionization: $\mathrm{e} + \mathrm{X} \rightarrow 2\mathrm{e} + \mathrm{X}^+$
  - recombination: $2\mathrm{e} + \mathrm{X}^+ \rightarrow \mathrm{e} + \mathrm{X}$
- inelastic collisions with other particles less frequent $\Rightarrow$ neglected

Microscopic processes

interaction with radiation

- excitation: $\nu + \mathrm{X} \rightarrow \mathrm{X}^*$
- deexcitation:
  - spontaneous: $\mathrm{X}^* \rightarrow \nu + \mathrm{X}$
  - stimulated: $\nu + \mathrm{X}^* \rightarrow 2\nu + \mathrm{X}$
- ionization: $\nu + \mathrm{X} \rightarrow \mathrm{X}^+ + \mathrm{e}$
- recombination:
  - spontaneous: $\mathrm{e} + \mathrm{X}^+ \rightarrow \nu + \mathrm{X}$
  - stimulated: $\nu + \mathrm{e} + \mathrm{X}^+ \rightarrow 2\nu + \mathrm{X}$

Microscopic processes

interaction with radiation

- excitation: $\nu + \mathrm{X} \rightarrow \mathrm{X}^*$
- deexcitation:
  - spontaneous: $\mathrm{X}^* \rightarrow \nu + \mathrm{X}$
  - stimulated: $\nu + \mathrm{X}^* \rightarrow 2\nu + \mathrm{X}$
- ionization: $\nu + \mathrm{X} \rightarrow \mathrm{X}^+ + \mathrm{e}$
  - autoionization: $\nu + \mathrm{X} \rightarrow \mathrm{X}^{**} \rightarrow \mathrm{X}^+ + \mathrm{e}$
  - Auger ionization: $\nu + \mathrm{X} \rightarrow \mathrm{X}^{+*}$
- recombination:
  - spontaneous: $\mathrm{e} + \mathrm{X}^+ \rightarrow \nu + \mathrm{X}$
  - stimulated: $\nu + \mathrm{e} + \mathrm{X}^+ \rightarrow 2\nu + \mathrm{X}$
  - dielectronic recombination: $\mathrm{X}^+ + \mathrm{e} \rightarrow \mathrm{X}^{**} \rightarrow \nu + \mathrm{X}$

Microscopic processes

- free-free transitions $\nu + \mathrm{e} + \mathrm{X} \leftrightarrow \mathrm{e} + \mathrm{X}$
- electron scattering
  - free (Compton, Thomson): $\nu + \mathrm{e} \rightarrow \nu + \mathrm{e}$
  - bound (Rayleigh): $\nu + \mathrm{X} \rightarrow \nu + \mathrm{X}$

LTE and NLTE

- silent background – maxwellian velocity distribution
  - inelastic collisions (collisional ionizations and excitations) destroy equilibrium velocity distribution
  - equilibrium is maintained by elastic collisions
  - $t_{\mathrm{relaxation}} \ll t_{\mathrm{inelastic\ collisions}}$ for most situations
  - exceptions: medium with few electrons

in the following we assume maxwellian (i.e. equilibrium) velocity distribution for all particles
radiation field is not in equilibrium – determined via the solution of the radiative transfer equation

LTE versus NLTE

maxwellian velocity distribution

- processes entering the game
  - collisional excitation and ionization (E)
  - radiative recombination (E)
  - free-free transitions (E)
  - photoionization
  - radiative excitation and deexcitation
  - elastic collisions (E)
  - Auger ionization
  - autoionization
  - dielectronic recombination (E)

LTE versus NLTE

detailed balance

- rate of each process is balanced by rate of the reverse process
- maxwellian distribution of electrons $\Rightarrow$ collisional processes in detailed balance
- radiative transitions in detailed balance only for Planck radiation field
- if $J_\nu \neq B_\nu \Rightarrow$ LTE not acceptable approximation

Atomic level populations

Boltzmann excitation formula

in equilibrium – Boltzmann distribution function

$$\frac{n^*_{ij}}{n^*_{0i}} = \frac{g_{ij}}{g_{0j}} e^{-\frac{\chi_i}{kT}}$$

$$\frac{n^*_{mj}}{n^*_{lj}} = \frac{g_{mj}}{g_{lj}} e^{-\frac{\chi_{mj}-\chi_{lj}}{kT}} = \frac{g_{mj}}{g_{lj}} e^{-\frac{h\nu_{lj;mj}}{kT}}$$

Boltzmann excitation formula

in equilibrium – Boltzmann distribution function

$$\frac{n^*_{ij}}{n^*_{0i}} = \frac{g_{ij}}{g_{0j}} e^{-\frac{\chi_i}{kT}}$$

summing over all levels

$$N^*_j = \sum_i n^*_{ij} = \frac{n^*_{0j}}{g_{0j}} \sum g_{ij} e^{-\frac{\chi_i}{kT}} = \frac{n^*_{0j}}{g_{0j}} U_j(T)$$

where

$$U_j(T) = \sum g_{ij} e^{-\frac{\chi_{ij}}{kT}}$$

partition function

Lowering of ionization potential

$$U_j(T) = \sum g_{ij} e^{-\frac{\chi_{ij}}{kT}}$$

partition function diverges, but not all states exist

$$\Delta\chi \approx 3 \cdot 10^{-8} Z \sqrt{\frac{n_e}{T}}\ \mathrm{eV}$$

Occupation probabilities of energy levels w_i

Saha ionization formula

Generalized ionization formula

$$\frac{n^*_{0,j+1}}{n^*_{0,j}} = \frac{2g_{0,j+1}}{g_{0,j}} \frac{1}{n_e} \left(\frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}} e^{-\frac{\chi_{I,j}}{kT}}$$

definition of LTE populations

$$n^*_{ij} = n_{0,j+1} n_e \frac{g_{ij}}{g_{0,j+1}} \underbrace{\frac{1}{2} \left(\frac{h^2}{2\pi mk} \right)^{\frac{3}{2}}}_{C_I = 2.07 \cdot 10^{-16}} T^{-\frac{3}{2}} e^{-\frac{\chi_{Ij} - \chi_{ij}}{kT}} = n_{0,j+1} n_e \Phi_{ij}(T)$$

Summing all levels of lower and upper ionization states

$$\frac{N^*_j}{N^*_{j+1}} = n_e \frac{U_j(T)}{U_{j+1}(T)} C_I T^{-\frac{3}{2}} e^{\frac{\chi_{Ij}}{kT}} = n_e \widetilde{\Phi}_j(T)$$

Thermodynamic equilibrium

Kirchhoff law

$$\eta(\nu) = \kappa(\nu) I(\nu) \tag{3}$$

for the case of equilibrium $I = B$ – Kirchhoff-Planck relation

$$\eta^*(\nu) = \kappa^*(\nu) B(\nu) \tag{4}$$

Local thermodynamic equilibrium radiation may differ from the thermodynamic equilibrium

$$\eta^{\mathrm{th}}(\nu) = \kappa^*(\nu) B_\nu[T(\vec{r}, t)] \tag{5}$$

simplification of calculations

often used, successful

often leads to satisfactory results

inconsistent – seriously fails

Conditions for LTE

Detailed balance

rate of each process = rate of the reverse process

collision build equilibrium, if the velocity distribution is maxwellian

maxwellian velocity distribution of electrons $\Rightarrow$ collisional excitation and a deexcitation in detailed balance

towards equilibrium also photorecombination and free-free transitions – basically they are collisions

radiative transitions – in detailed balance only for isotropic intensity with Planckian distribution in frequencies

LTE is valid in deep layers of stellar atmospheres

Electron velocity distribution

ionization and excitation affect equilibrium distribution – inelastic collisions

in stellar atmospheres holds

relaxation time $\ll$ time between successive recombinations $H^+ + e \rightarrow H$

$H + e \rightarrow H^-$

$\ll$ time between sucessive excitations

(typical energy 1eV, ionization about 10eV, only 10^{-5} elektrons may excite)

differences from Maxwellian distribution may be there, where there is less electrons (Sun)

then it is necessary to solve the kinetic equation for electrons

Equations of statistical equilibrium

change of the state i of each element

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}) = \sum_{j \neq i} (n_j P_{ji} - n_i P_{ij})$$

P_{ij} – transition probability from the level i to the level j

Equations of statistical equilibrium

change of the state i of each element

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P_{ij} – transition probability from the level i to the level j
continuity equation for element k,

$$\frac{\partial N_k}{\partial t} + \nabla \cdot (N_k \vec{v}) = 0.$$

gas continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0.$$

Equations of statistical equilibrium

change of the state i of each element

$$\cancel{\frac{\partial n_i}{\partial t}} + \nabla \cdot (n_i \vec{v}) = \sum_{j \neq i} (n_j P_{ji} - n_i P_{ij})$$

P_{ij} – transition probability from the level i to the level j

- stationary state or negligible changes with time – without $\partial / \partial t$

Equations of statistical equilibrium

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- static state ($\vec{v} = 0$) or negligible advection (used in stellar winds) – also without ∇

Equations of statistical equilibrium

change of the state i of each element

$$0 = \sum_{j \neq i} (n_j P_{ji} - n_i P_{ij})$$

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Equations of statistical equilibrium

change of the state i of each element

$$0 = \sum_{j \neq i} (n_j P_{ji} - n_i P_{ij})$$

P_{ij} – transition probability from the level i to the level j

- $P_{ij} = R_{ij} + C_{ij}$
- R_{ij} – radiative rates
- C_{ij} – collisional rates

Equations of statistical equilibrium

change of the state i of each element

$$0 = \sum_{j \neq i} (n_j P_{ji} - n_i P_{ij})$$

P_{ij} – transition probability from the level i to the level j

- detailed balance is for $n_j P_{ji} = n_i P_{ij}$, for $\forall i, j$
- equilibrium populations n_i^*,

Equilibrium level populations

- n_i^* – LTE level population
- departure coefficients $b_i = \frac{n_i}{n_i^*}$, for LTE $b_i = 1$

definition of $n_{i,j}^*$ (level i of ion j)

1. population with the assumption of LTE
2. $n_{i,j}^* = n_{0,j+1} n_e \frac{g_{ij}}{g_{0,j+1}} \frac{1}{2} \left(\frac{h^2}{2\pi mkT} \right)^{\frac{3}{2}} e^{-\frac{\chi_{Ij} - \chi_{ij}}{kT}}$

   $n_{0,j+1}$ – ***actual*** population of the ground level of the next higher ion

Radiative rates – bound-free

photoionization from the state i:
amount of absorbed energy: $4\pi J_\nu \alpha_{ik}(\nu)\,\mathrm{d}\nu$
number of photoionization is obtained dividing by $h\nu$ and integrating from 0 to ∞:

$$n_i R_{ik} = n_i 4\pi \int_{\nu_0}^{\infty} \frac{\alpha_{ik}}{h\nu} J_\nu \,\mathrm{d}\nu$$

Radiative rates – free-bound

photorecombination – collisional process

for TE $\Rightarrow$ detailed balance and $J_\nu = B_\nu$

$$n_k^* R_{ki}^* = n_i^* R_{ik}^* = n_i^* 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} B_\nu \,\mathrm{d}\nu$$

Radiative rates – free-bound

photorecombination – collisional process

for TE $\Rightarrow$ detailed balance and $J_\nu = B_\nu$

$$n_k^* R_{ki}^* = n_i^* R_{ik}^* = n_i^* 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} B_\nu \,\mathrm{d}\nu$$

$$n_k^* R_{ki}^* = n_i^* 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} \left[B_\nu \left(1 - e^{-\frac{h\nu}{kT}} \right) + B_\nu e^{-\frac{h\nu}{kT}} \right] \,\mathrm{d}\nu$$

Radiative rates – free-bound

photorecombination – collisional process

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$$n_k^* R_{ki}^* = n_i^* 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} \left[\frac{2h\nu^3}{c^2} + B_\nu\right] e^{-\frac{h\nu}{kT}} \,\mathrm{d}\nu$$

Radiative rates – free-bound

photorecombination – collisional process

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per ion

$$R_{ki}^* = \frac{n_i^*}{n_k^*} 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} \left[\frac{2h\nu^3}{c^2} + B_\nu \right] e^{-\frac{h\nu}{kT}} \,\mathrm{d}\nu$$

Radiative rates – free-bound

photorecombination – collisional process

for TE $\Rightarrow$ detailed balance and $J_\nu = B_\nu$

per ion

$$R^*_{ki} = \frac{n^*_i}{n^*_k} 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} \left[\frac{2h\nu^3}{c^2} + B_\nu \right] e^{-\frac{h\nu}{kT}} \, \mathrm{d}\nu$$

Radiative rates – free-bound

photorecombination – collisional process

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valid also outside TE

Radiative rates – free-bound

photorecombination – collisional process

for TE $\Rightarrow$ detailed balance and $J_\nu = B_\nu$

per ion

$$R_{ki} = R^*_{ki} = \frac{n^*_i}{n^*_k} 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} \left[\frac{2h\nu^3}{c^2} + B_\nu\right] e^{-\frac{h\nu}{kT}}\,\mathrm{d}\nu$$

valid also outside TE

replace $B_\nu \rightarrow J_\nu$ and multiply by actual number of ions n_k

$$n_k R_{ki} = n_k \frac{n^*_i}{n^*_k} 4\pi \int_0^\infty \frac{\alpha_{ik}(\nu)}{h\nu} \left[\frac{2h\nu^3}{c^2} + J_\nu\right] e^{-\frac{h\nu}{kT}}\,\mathrm{d}\nu$$

Radiative rates – bound-bound (up)

number of transitions $i \to j$ caused by intensity I in $\mathrm{d}\nu\,\mathrm{d}\omega$

$$n_i B_{ij} \phi_\nu I_\nu \,\mathrm{d}\nu \frac{\mathrm{d}\omega}{4\pi} = n_i B_{ij} \phi_\nu J_\nu \,\mathrm{d}\nu$$

total number of absorptions by integration over the profile

$$n_i R_{ij} = n_i B_{ij} \int \phi_\nu J_\nu \,\mathrm{d}\nu = n_i 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} J_\nu \,\mathrm{d}\nu$$

since $\alpha_\nu = \frac{h\nu}{4\pi} B_{ij} \phi_\nu$

Radiative rates – bound-bound (down)

number of stimulated emissions

$$n_j R_{ji}^{\mathrm{stim}} = n_j B_{ji} \int \phi_\nu J_\nu \,\mathrm{d}\nu = n_i 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} J_\nu \,\mathrm{d}\nu$$

Radiative rates – bound-bound (down)

number of stimulated emissions

$$n_j R_{ji}^{\mathrm{stim}} = n_j B_{ji} \int \phi_\nu J_\nu \,\mathrm{d}\nu = n_i 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} J_\nu \,\mathrm{d}\nu$$

number of spontaneous emissions

$$n_j R_{ji}^{\mathrm{spont}} = n_j A_{ji} = n_j \frac{2h\nu_{ij}^3}{c^2} B_{ji} = n_j \frac{g_i}{g_j} \frac{2h\nu_{ij}^3}{c^2} B_{ij} = n_j \frac{g_i}{g_j} \frac{4\pi}{h\nu_{ij}} \frac{2h\nu_{ij}^3}{c^2} \alpha_{ij}$$

Radiative rates – bound-bound (down)

number of stimulated emissions

$$n_j R_{ji}^{\rm stim} = n_j B_{ji} \int \phi_\nu J_\nu \, \mathrm{d}\nu = n_i 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} J_\nu \, \mathrm{d}\nu$$

number of spontaneous emissions

$$n_j R_{ji}^{\rm spont} = n_j A_{ji} = n_j \frac{2h\nu_{ij}^3}{c^2} B_{ji} = n_j \frac{g_i}{g_j} \frac{2h\nu_{ij}^3}{c^2} B_{ij} = n_j \frac{g_i}{g_j} \frac{4\pi}{h\nu_{ij}} \frac{2h\nu_{ij}^3}{c^2} \alpha_{ij}$$

total number of emissions

$$n_j R_{ji} = n_j \left(A_{ji} + B_{ji} \int \phi_\nu J_\nu \, \mathrm{d}\nu \right)$$

Radiative rates – bound-bound (down)

total number of emissions

$$n_j R_{ji} = n_j \left(A_{ji} + B_{ji} \int \phi_\nu J_\nu \, \mathrm{d}\nu \right)$$

Radiative rates – bound-bound (down)

total number of emissions

$$n_j R_{ji} = n_j \left(A_{ji} + B_{ji} \int \phi_\nu J_\nu \, \mathrm{d}\nu \right)$$
$$= n_j \frac{4\pi}{h\nu_{ij}} \frac{g_i}{g_j} \alpha_{ij} \left[\frac{2h\nu_{ij}^3}{c^2} + \int \phi_\nu J_\nu \, \mathrm{d}\nu \right]$$

Radiative rates – bound-bound (down)

total number of emissions

$$n_j R_{ji} = n_j \left(A_{ji} + B_{ji} \int \phi_\nu J_\nu \,\mathrm{d}\nu \right)$$
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the Boltzmann equation

$$\frac{n_i^*}{n_j^*} = \frac{g_i}{g_j} \exp\left(\frac{h\nu_{ij}}{kT} \right)$$

Radiative rates – bound-bound (down)

total number of emissions

$$n_j R_{ji} = n_j \left(A_{ji} + B_{ji} \int \phi_\nu J_\nu \,\mathrm{d}\nu \right)$$

$$= n_j \frac{4\pi}{h\nu_{ij}} \frac{g_i}{g_j} \alpha_{ij} \left[\frac{2h\nu_{ij}^3}{c^2} + \int \phi_\nu J_\nu \,\mathrm{d}\nu \right]$$

$$= n_j \frac{n_i^*}{n_j^*} 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} \left[\frac{2h\nu^3}{c^2} + J_\nu \right] e^{-\frac{h\nu}{kT}} \,\mathrm{d}\nu$$

the Boltzmann equation

$$\frac{n_i^*}{n_j^*} = \frac{g_i}{g_j} \exp\left(\frac{h\nu_{ij}}{kT} \right)$$

Radiative rates – total

upward $i \rightarrow l$:

$$n_i R_{il} = n_i 4\pi \int \frac{\alpha_{il}(\nu)}{h\nu} J_\nu \, \mathrm{d}\nu$$

downward $l \rightarrow i$:

$$n_l R_{li} = n_l \frac{n_i^*}{n_l^*} 4\pi \int \frac{\alpha_{il}(\nu)}{h\nu} \left[\frac{2h\nu^3}{c^2} + J_\nu \right] e^{-\frac{h\nu}{kT}} \, \mathrm{d}\nu$$

Collisional rates

sufficient to consider only electrons, because $v_{\mathrm{th,e}}/v_{\mathrm{th,i}} \approx 43\sqrt{A}$

rate up

$$n_i C_{ij} = n_i n_e \int_{v_0}^{\infty} \sigma_{ij}(v) f(v) v \, \mathrm{d}v = n_i n_e q_{ij}(T)$$

$\sigma_{ij}(v)$ – total cross section of the transition $i \to j$

rate down from the detailed balance $n_j^* C_{ji} = n_i^* C_{ij}$

$$n_j C_{ji} = n_j \left(\frac{n_i^*}{n_j^*} \right) C_{ij} = n_j \left(\frac{n_i^*}{n_j^*} \right) n_e q_{ij}(T)$$

System of statistical equilibrium equations

$\forall$ level

$$n_i \sum_l \left(R_{il} + C_{il}\right) + \sum_l n_l \left(R_{li} + C_{li}\right) = 0$$

linearly dependent equations
supplementary equations

- charge conservation $\sum_k \sum_j j N_{jk} + n_p = n_e$
- particle number conservation $\sum_k \sum_j N_{jk} = N_N$
- abundance equation $\sum_j N_{jk} = \frac{\alpha_k}{\alpha_H} \sum_j N_{jH}$

Radiative force

equation of motion

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla \cdot \mathsf{P} + \rho \vec{g} = -\nabla \cdot \left(\mathsf{P}_g + \mathsf{P}_R \right) + \rho \vec{g}$$

Radiative force

equation of motion

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla \cdot \mathsf{P} + \rho \vec{g} = -\nabla \cdot (\mathsf{P}_g + \mathsf{P}_R) + \rho \vec{g}$$

2nd moment radiative transfer equation

$$\frac{1}{c^2} \frac{\partial \vec{\mathcal{F}}_\nu}{\partial t} + \nabla \cdot \mathsf{P}_R(\nu) = \frac{1}{c} \oint \vec{n} \left[\eta_\nu(\vec{n}) - \chi_\nu(\vec{n}) I_\nu(\vec{n}) \right] \mathrm{d}\omega$$

for static medium and isotropic χ and η

$$\nabla \cdot \mathsf{P}_R(\nu) = -\frac{1}{c} \chi_\nu \vec{\mathcal{F}}_\nu$$

Radiative force

equation of motion

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla \cdot \mathsf{P} + \rho \vec{g} = -\nabla \cdot (\mathsf{P}_g + \mathsf{P}_R) + \rho \vec{g}$$

2nd moment radiative transfer equation for static medium and isotropic χ and η

$$\nabla \cdot \mathsf{P}_R(\nu) = -\frac{1}{c}\,\chi_\nu\,\vec{\mathcal{F}}_\nu$$

Radiative force

equation of motion

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla \cdot \mathsf{P} + \rho \vec{g} = -\nabla \cdot (\mathsf{P}_g + \mathsf{P}_R) + \rho \vec{g}$$

2nd moment radiative transfer equation for static medium and isotropic χ and η

$$\nabla \cdot \mathsf{P}_R(\nu) = -\frac{1}{c}\, \chi_\nu\, \vec{\mathcal{F}}_\nu$$

integrate over ν

$$\nabla \cdot \mathsf{P}_R = \int_0^\infty \nabla \cdot \mathsf{P}_R(\nu)\, \mathrm{d}\nu = -\int_0^\infty \frac{1}{c}\, \chi_\nu\, \vec{\mathcal{F}}_\nu\, \mathrm{d}\nu$$

Radiative force

equation of motion

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \underbrace{\int_0^\infty \frac{1}{c}\,\chi_\nu\,\vec{\mathcal{F}}_\nu\,\mathrm{d}\nu}_{} + \rho \vec{g}$$

Radiative force

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \underbrace{\int_0^\infty \frac{1}{c} \chi_\nu \vec{\mathcal{F}}_\nu \, \mathrm{d}\nu}{} + \rho \vec{g}$$

- depends on radiation flux $\vec{\mathcal{F}}$
- depends on opacity χ_ν

Radiative force

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \underbrace{\int_0^\infty \frac{1}{c} \chi_\nu \vec{\mathcal{F}}_\nu \, \mathrm{d}\nu}_{} + \rho \vec{g}$$

- depends on radiation flux $\vec{\mathcal{F}}$
- depends on opacity χ_ν
- continuum radiative force (electron scattering)
- electron scattering radiative force
- line radiative force
  - global effects (radiaively driven stellar winds)
  - selective effects (radiative diffusion, radiative levitation, CP stars, multicomponent effects in winds)

Classical model atmospheres

assumptions

- geometry – plane-parallel, spherically symmetric (horizontally homogeneous)
- stationarity – $\vec{v} = 0$, $\frac{\partial}{\partial t} = 0$
- energy equilibrium
  - radiative equilibrium (for static medium)

Opacity in models

$$\chi_\nu = \sum_i \sum_{j \neq i} \left[n_i - \frac{g_i}{g_j} n_j \right] \alpha_{ij}(\nu) + \sum_i \left(n_i - n_i^* e^{-\frac{h\nu}{kT}} \right) \alpha_{ik}(\nu) +$$

$$\sum_k n_e n_k \alpha_{kk}(\nu, T) \left(1 - e^{-\frac{h\nu}{kT}} \right) + n_e \sigma_e$$

$$\eta_\nu = \frac{2h\nu^3}{c^2} \left[\sum_i \sum_{j \neq i} n_j \frac{g_i}{g_j} \alpha_{ij}(\nu) + \sum_i n_i^* \alpha_{ik}(\nu) e^{-\frac{h\nu}{kT}} + \right.$$

$$\left. \sum_k n_e n_k \alpha_{kk}(\nu, T) e^{-\frac{h\nu}{kT}} \right]$$

chemical composition (free parameter) H, He, . . .

bound-free transitions	$T < \odot$	H^-
	A	H
	B	H + HeI
	O	HeII
free-free transitions	M	H_2^-
	$\odot$	H^-
	A	H
	O	H, HeI, HeII
scattering	O	electrons
	G, K	Rayleigh scattering
bound-bound transitions	O	H, HeI, HeII
	A	H
	$\odot$	metals – neutral and ionized
	late	CN, CO, H_2O

Line blanketing

opacity distribution function resampling of opacities

- lowers the number of frequency points (a lot)
- costs
  - difficulties with opacity variable with depth
  - difficulties with inclusion of velocity gradients
  - difficulties in NLTE calculations

opacity sampling randomly chosen frequencies – neccesary to be enough of them

superlines and superlevels

Hydrostatic equilibrium

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \rho \vec{g}$$

for static medium

$$\nabla p = \rho \vec{g},$$

where $p = p_g + p_R$.
column mass depth $\mathrm{d}m = -\rho\,\mathrm{d}z$ (for spherical symmetry $\mathrm{d}m = -\rho \frac{R^2}{r^2}\,\mathrm{d}r$)

Hydrostatic equilibrium

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \rho \vec{g}$$

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column mass depth $\mathrm{d}m = -\rho\,\mathrm{d}z$ (for spherical symmetry $\mathrm{d}m = -\rho \frac{R^2}{r^2}\,\mathrm{d}r$)

for one-dimensional atmosphere

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -g\rho$$

Hydrostatic equilibrium

for one-dimensional atmosphere

$$-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\mathrm{d}p}{\mathrm{d}m} = \frac{\mathrm{d}p_g}{\mathrm{d}m} + \frac{\mathrm{d}p_R}{\mathrm{d}m} + g$$

$$\rho\frac{\mathrm{d}p_R}{\mathrm{d}m} = \frac{\mathrm{d}p_R}{\mathrm{d}z} = -\frac{4\pi}{c}\int_0^\infty \chi_\nu H_\nu \,\mathrm{d}\nu$$

$$\frac{\mathrm{d}p_g}{\mathrm{d}m} = g - \frac{4\pi}{c}\int_0^\infty \frac{\chi_\nu}{\rho} H_\nu \,\mathrm{d}\nu$$

+ upper boundary condition

$$N_1 k T_1 = m_1 g - \frac{4\pi}{c}\int_0^\infty K_{\nu 1}\,\mathrm{d}\nu$$

Energy equilibrium

energy transfer in the atmosphere

- by radiation (always)
- by convection – cool stars (F, G, K, M, . . .)

$$\nabla \cdot \vec{\mathcal{F}} = \nabla \cdot (\vec{\mathcal{F}}_R + \vec{\mathcal{F}}_C) = 0$$

Radiative equilibrium

$$4\pi \int_0^\infty (\eta_\nu - \chi_\nu J_\nu)\, \mathrm{d}\nu = 0$$

or

$$\nabla \cdot \vec{\mathcal{F}} = 0 \qquad \Rightarrow \qquad \vec{\mathcal{F}} = \text{const}$$

- integral form $\rightarrow$ does not guarantee flux conservation, stable
- differential form $\rightarrow$ guarantees flux conservation, for small τ unstable
- superposed form $\rightarrow$ $\alpha I + \beta D = 0$

temperature correction methods (Unsöld & Lucy, Avrett & Krook), various usage of mean opacities
linearized solution of RTE+RE

Convection

convective (in)stability

obrázek

element at a point P we move to a point P' for Δr

- slow motion $\rightarrow$ element in equilibrium with surroundings
- no energy exchange with surroundings (adiabatic process)

change of the density $(\Delta\rho)_E = \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_A \Delta r$ at P' (pressure $p = p'$)
ρ_R – density in the surroundings of P'

- $\rho_E < \rho_R$ – element will continue upwards
- $\rho_E > \rho_R$ – element will return back – stability

Convection

condition of instability

$$\nabla_R = \left(\frac{\mathrm{d} \ln T}{\mathrm{d} \ln p} \right)_R > \frac{\gamma - 1}{\gamma} = \nabla_A$$

$\gamma = c_P / c_V$

Convection

condition of instability

$$\nabla_R = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln p}\right)_R > \frac{\gamma - 1}{\gamma} = \nabla_A$$

$\gamma = c_P/c_V$

for non-ideal gases $\gamma \to \Gamma$ – Chandrasekhar adiabatic index
typical values:

- $\Gamma = \frac{5}{3}$ – monoatomic gas
- $\Gamma = \frac{4}{3}$ – radiation
- $\Gamma \sim 1.1$ – in the hydrogen ionization region

Convection

for non-ideal gases $\gamma \to \Gamma$ – Chandrasekhar adiabatic index
typical values:

- $\Gamma = \frac{5}{3}$ – monoatomic gas
- $\Gamma = \frac{4}{3}$ – radiation
- $\Gamma \sim 1.1$ – in the hydrogen ionization region

where is convection in the atmosphere

- for hot stars weak conection in ionization region of He a HeI
- A stars $\tau \sim 0.2$
- F and cooler stars – convective zone increases and dominates
- M stars – determines the atmospheric structure

Modeling of convection

turbulent medium $\Rightarrow$ mathematical description complicated

mixing length theory

$$\pi F_c = \frac{1}{2}\rho c_p T v \frac{l}{H_p}\left[\frac{\gamma}{1+\gamma}\left(\nabla - \nabla_{\rm ad}\right)\right]$$

approximation of the turbulent medium by one eddy (lower energy)
free parameter $l = \alpha H$ – we can model everything

Modeling of convection

turbulent convection model – Canuto & Mazzitelli

$$F_c = \frac{kT}{H_p}\left(\nabla - \nabla_{\rm ad}\right)\Phi \qquad K = \frac{16\sigma T^3}{3\bar{\chi}_R}$$

$$H_p = \frac{p}{\rho g} \qquad \nabla = \left(\frac{\partial \ln T}{\partial \ln p}\right)$$

$$\Phi = 24.868\Sigma^{0.14972}\left[(1 + 0.097666\Sigma)^{0.18931} - 1\right]^{1.8503}$$

$$\Sigma = 4A^2\left(\nabla - \nabla_{\rm ad}\right) \qquad A = \frac{l^2 c_p \rho}{9k}\sqrt{\frac{g}{H_p}}$$

no free parameter, because we take $l = H_p$

Modeling of convection

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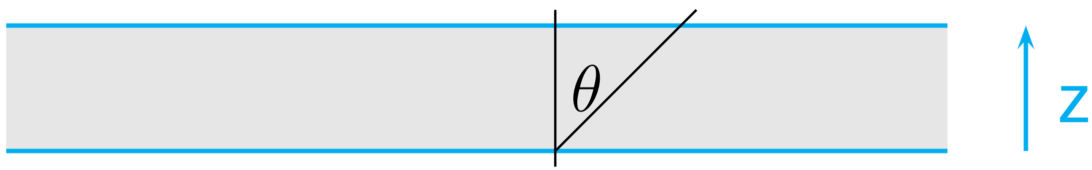
no free parameter, because we take $l = H_p$

$$\Phi_{\rm MLT} = \frac{9}{8}\frac{\left(\sqrt{1+\Sigma}-1\right)^3}{\Sigma}$$

Model atmosphere construction

simplified situation: 1D static plane-parallel atmosphere

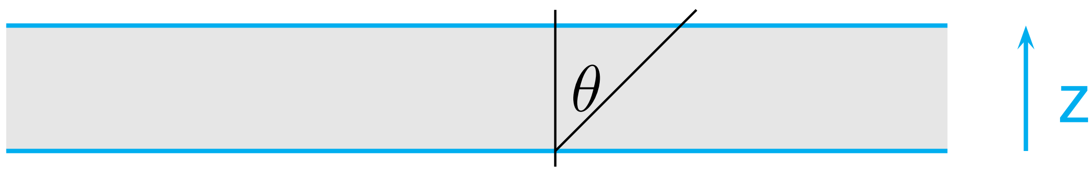


we take into account

- radiation from all directions (one angle variable)
- full frequency spectrum

Model atmosphere construction

simplified situation: 1D static plane-parallel atmosphere



we take into account

- radiation from all directions (one angle variable)
- full frequency spectrum

solution of

- radiative transfer equation ($I_{\mu\nu}$)
- equation of radiative equilibrium (T)
- equation of hydrostatic equilibrium (ρ)
- equations of statistical equilibrium (n_i)

Model atmosphere construction

- radiative transfer equation ($I_{\mu\nu}$)
- equation of radiative equilibrium (T)
- equation of hydrostatic equilibrium (ρ)
- equations of statistical equilibrium (n_i)

Model atmosphere construction

- radiative transfer equation ($I_{\mu\nu}$)

$$\mu \frac{\mathrm{d}I_{\mu\nu}(z)}{\mathrm{d}z} = \eta_\nu(z) - \chi_\nu(z) I_{\mu\nu}(z)$$

- equation of radiative equilibrium (T)
- equation of hydrostatic equilibrium (ρ)
- equations of statistical equilibrium (n_i)

Model atmosphere construction

- radiative transfer equation ($I_{\mu\nu}$)

$$\mu \frac{\mathrm{d}I_{\mu\nu}(z)}{\mathrm{d}z} = \eta_\nu(z) - \chi_\nu(z) I_{\mu\nu}(z)$$

- equation of radiative equilibrium (T)

$$4\pi \int_0^\infty (\chi_\nu J_\nu - \eta_\nu)\ \mathrm{d}\nu = 0$$

- equation of hydrostatic equilibrium (ρ)
- equations of statistical equilibrium (n_i)

Model atmosphere construction

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$$\frac{\mathrm{d}p}{\mathrm{d}m} = g - \frac{4\pi}{c} \int_0^\infty \frac{\chi_\nu}{\rho} H_\nu \, \mathrm{d}\nu$$

- equations of statistical equilibrium (n_i)

Model atmosphere construction

- radiative transfer equation ($I_{\mu\nu}$)

$$\mu \frac{\mathrm{d}I_{\mu\nu}(z)}{\mathrm{d}z} = \eta_\nu(z) - \chi_\nu(z) I_{\mu\nu}(z)$$

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$$\frac{\mathrm{d}p}{\mathrm{d}m} = g - \frac{4\pi}{c} \int_0^\infty \frac{\chi_\nu}{\rho} H_\nu \,\mathrm{d}\nu$$

- equations of statistical equilibrium (n_i)

$$n_i \sum_l (R_{il} + C_{il}) + \sum_l n_l (R_{li} + C_{li}) = 0$$

Model atmosphere construction

calculation of opacity and emissivity

$$\chi_\nu = \sum_i \sum_{j\neq i} \left[n_i - \frac{g_i}{g_j} n_j \right] \alpha_{ij}(\nu) + \sum_i \left(n_i - n_i^* e^{-\frac{h\nu}{kT}} \right) \alpha_{ik}(\nu) +$$

$$\sum_k n_e n_k \alpha_{kk}(\nu, T) \left(1 - e^{-\frac{h\nu}{kT}} \right) + n_e \sigma_e$$

$$\eta_\nu = \frac{2h\nu^3}{c^2} \left[\sum_i \sum_{j\neq i} n_j \frac{g_i}{g_j} \alpha_{ij}(\nu) + \sum_i n_i^* \alpha_{ik}(\nu) e^{-\frac{h\nu}{kT}} + \right.$$

$$\left. \sum_k n_e n_k \alpha_{kk}(\nu, T) e^{-\frac{h\nu}{kT}} \right]$$

Model atmosphere construction

calculation of populations (n_i)

LTE: $n_i = n_i\left(N, T\right)$

$$\frac{n_m^*}{n_m^*} = \frac{g_m}{g_l} e^{-\frac{h\nu_{lm}}{kT}}$$

$$\frac{N_j^*}{N_{j+1}^*} = n_e \frac{U_j(T)}{U_{j+1}(T)} C_I T^{-\frac{3}{2}} e^{\frac{\chi_{Ij}}{kT}} = n_e \tilde{\Phi}_j(T)$$

Model atmosphere construction

NLTE (SE): $n_i = n_i\left(N, T, J_\nu\right)$

$$n_i \sum_j \left(R_{ij} + C_{ij}\right) + \sum_j n_j \left(R_{ji} + C_{ji}\right) = 0$$

$$\sum_k \sum_j j N_{jk} = n_e$$

$$\sum_k \sum_j N_{jk} = N_N$$

$$\sum_j N_{jk} = \frac{\alpha_k}{\alpha_H} \sum_j N_{jH}$$

$$R_{ij} = 4\pi \int_0^\infty \alpha_{ij}(\nu) \frac{J_\nu}{h\nu}\, \mathrm{d}\nu$$

$$R_{ji} = 4\pi \left(\frac{n_i}{n_j}\right)^* \int_0^\infty \frac{\alpha_{ij}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu\right) e^{-\frac{h\nu}{kT}}\, \mathrm{d}\nu$$

$$C_{ij} = n_e q_{ij}(T)$$

$$C_{ji} = n_e \left(\frac{n_i}{n_j}\right)^* q_{ij}(T)$$

Complete linearization method

Auer & Mihalas 1969
ND depth points
NF frequency points

$$\vec{\psi}_d = (J_1, \ldots, J_{NF}, N, T, n_1, \ldots, n_{NL})$$

system of equations

$$\vec{f}_d(\vec{\psi}_d) = 0$$

$$\vec{\psi}_d \to \vec{\psi}_d^0 + \delta\vec{\psi}_d$$

$$\vec{f}_d(\vec{\psi}_d) = \vec{f}_d(\vec{\psi}_d^0 + \delta\vec{\psi}_d) = \vec{f}_d(\vec{\psi}_d^0) + \sum \frac{\partial \vec{f}_d}{\partial \vec{\psi}_{dj}} \delta\vec{\psi}_{dj} = 0$$

$\Rightarrow \delta\vec{\psi}_{dj} \to$ new values $\vec{\psi}_d$

Complete linearization method

taking into account, e.g., opacities

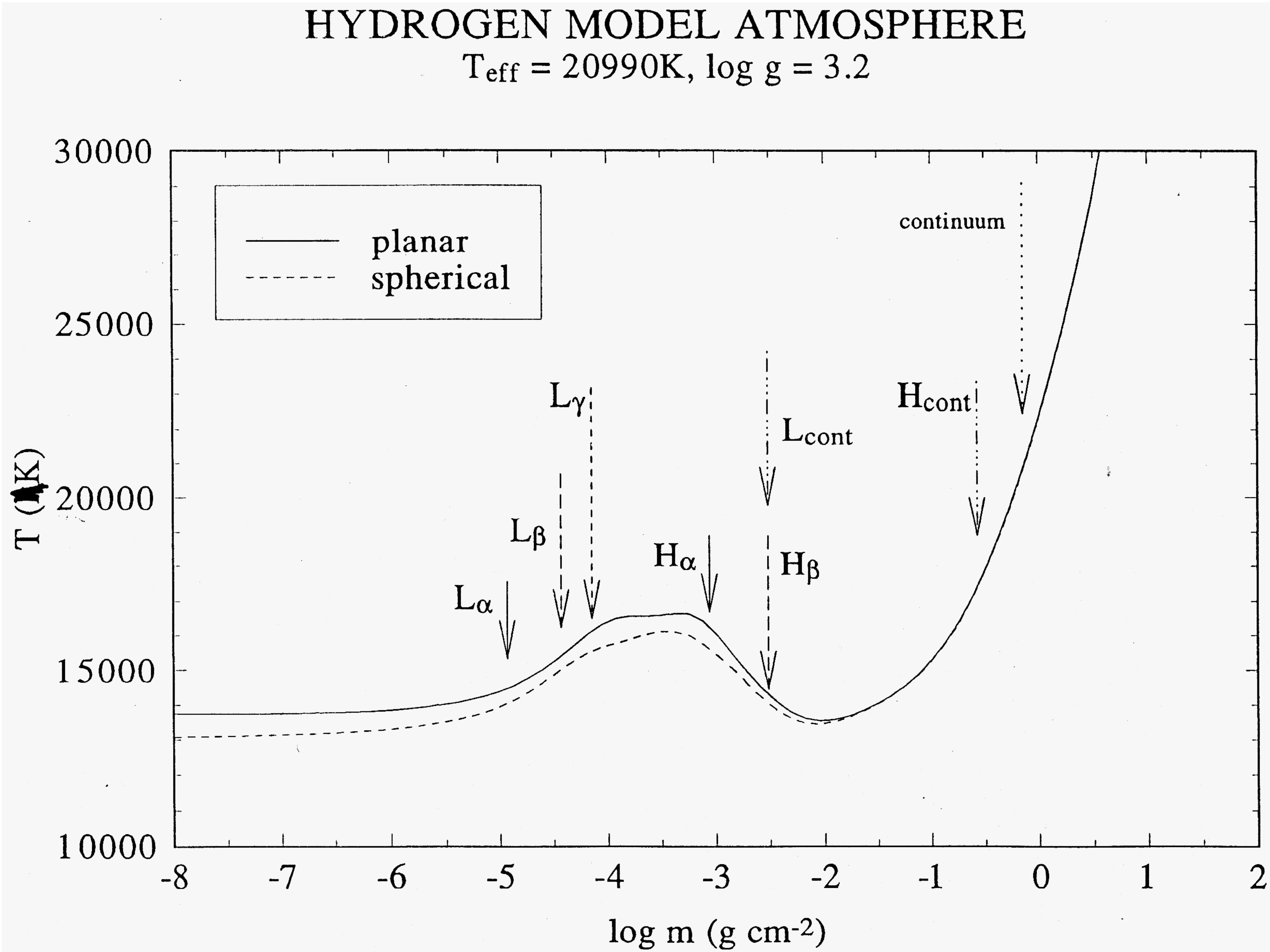
$$\delta\chi_{di} = \left.\frac{\partial\chi_i}{\partial T}\right|_d \delta T_d + \left.\frac{\partial\chi_i}{\partial n_e}\right|_d \delta(n_e)_d + \sum_{l=1}^{NL} \left.\frac{\partial\chi_i}{\partial n_l}\right|_d \delta(n_l)_d$$

employing ALI – iteration

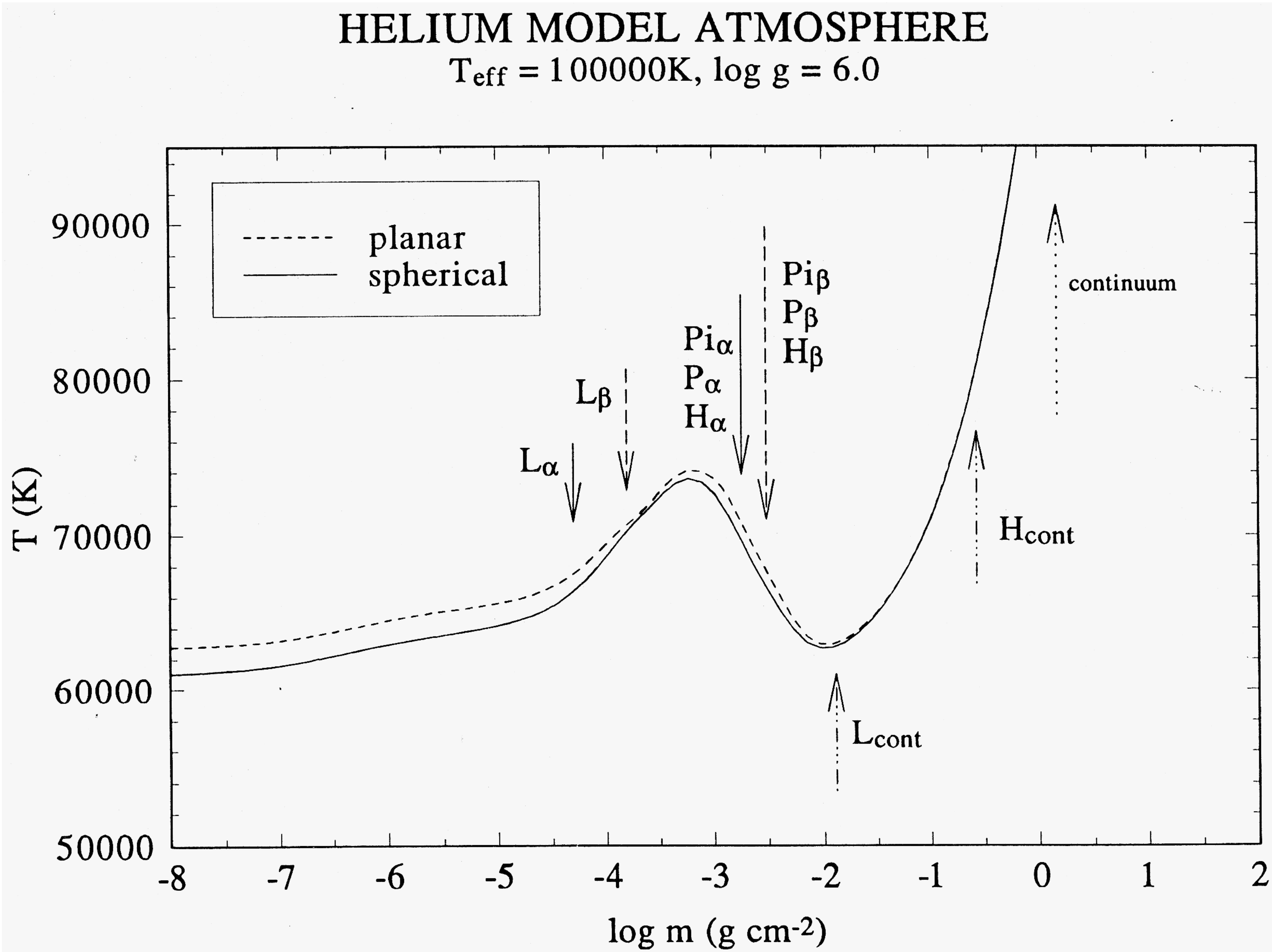
$$J_\nu^{(n)} = \Lambda^* S_\nu^{(n)} + \underbrace{\left(\Lambda - \Lambda^*\right) S_\nu^{(n-1)}}_{\Delta J_\nu^{(n-1)}}$$

we remove J from vector ψ

Line formation regions



Line formation regions



Stellar atmosphere codes

partial review – Sakhibullin

ATLAS – Kurucz – LTE models + line blanketing

TLUSTY – Hubeny – NLTE models + line blanketing `http://tlusty.gsfc.nasa.gov`

TMAP – Werner, Dreizler – Tübingen model atmosphere package, NLTE models + line blanketing using OS
`http://astro.uni-tuebingen.de/groups/stellar/#tmap`

PHOENIX – Hauschildt, Allard – NLTE models + moving atmospheres

CMFGEN – Hillier – NLTE models + moving atmospheres (includes most, the best code)

DETAIL

MARCS – Gustafsson – LTE models + line blanketing (for very cool stars)